Dynamical Cocycle Invariants of Knotted Surfaces

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1. Def of Quandles

2. A surface invariant

3. Crossed Modules

4. Quandle 3-cocycles from group cocycles

5. Conclusion

A QUANDLE is a set, Q, with a binary operation $*$ defined such that

- $a * a = a$

- $\forall a, b \in Q \; \exists ! c \in Q$ *s.t.* $a = c * b$

- $(a * b) * c = (a * c) * (b * c)$

Let G be a group. Define $a * b = bab^{-1}$. This is a quandle. Sp that $G \to$ Aut(A) — A an Abelian group. Write $(g, a) \mapsto ga$.

Def: A generalized quandle 3-cocycle is a function $\kappa : G \times G \times G \to A$ such that:

$$w\kappa_{x,y,z} + \kappa_{x*z,y*z,w} + ((y*z)*w)\kappa_{x,z,w} + \kappa_{y,z,w}$$

$$= ((x*(y*z))*w)\kappa_{y,z,w} + \kappa_{x*y,z,w}$$

$$+ (z*w)\kappa_{x,y,w} + \kappa_{x*w,y*w,z*w}$$

And

$$\kappa(x,x,y) = \kappa(x,y,y) = id.$$

THM (CEGS): There is a State-sum invariant for knotted surfaces using such cocycles. Cohomologous cocycles yield the same values.

Sketch:(a) The quandle cocycle is evaluated at source regions of triple points, taking the product over all triple points, and summing over all colorings.

(b) I lied. In fact, we take an arc from ∞ to the target region in a KSD. Then the labels on the regions that this arc crosses act on the value of the cocycle at these evaluations.

(c)Sp

$$\kappa = \delta\zeta(x, y, z)$$

$$= z\zeta_{x,y} + \zeta_{x*y,z} + (x * y) * z\zeta_{y,z}$$

$$-(y * z\zeta_{x,z} + \zeta_{y,z} + \zeta_{x*z,y*z}),$$

Crossed Modules (from Brown): N and E are groups. $\alpha : N \to E$ homom. $\exists$ An action of E on N : $(e, n) \mapsto {}^e n$ such that

$$\begin{aligned} {}^{\alpha(n)}n' &= nn'n^{-1} \quad (n, n' \in N) \\ \alpha({}^e n) &= e\alpha(n)e^{-1} \quad (e \in E,\ n \in N). \end{aligned}$$

$\Rightarrow$ im $\alpha \triangleleft E$, so set $G =$ coker α, $A =$ ker $\alpha \subset Z(N)$

$$0 \to A \xrightarrow{i} N \xrightarrow{\alpha} E \xrightarrow{\pi} G \to 1$$

is exact.

Let $s : G \to E$ be a section of π w/ $s(1) = 1$. ¿s =homom? $\exists f : G \times G \to$ ker π

$$s(x)s(y) = f(x, y)s(xy), \quad x, y \in G,$$

s.t.

$$f(x, y)f(xy, z) = {}^{s(x)}f(y, z)f(x, yz),$$

where ${}^{s(x)}f(y, z) = s(x)f(y, z)s(x)^{-1}$. Since ker $\pi =$ im α,

$\exists F : G \times G \to N$ ¿$F = 2 - cocycle$? $\exists c : G \times G \times G \to A$ s.t.

$^{s(x)}F(y,z)F(x,yz) = i(c(x,y,z))F(x,y)F(xy,z)$,

$c =$ group 3-cocycle.

Quandle structure: $* \; u * v = vuv^{-1}$ for all groups involved.

Lemma: Let $\phi : G \times G \to$ ker π

$$s(x) * s(y) = \phi(x,y)s(x*y), \quad x, y \in G.$$

Then $\phi(x,y) = f(y,x)f(yxy^{-1},y)^{-1}$.

Pf.

$$\begin{aligned} s(y)s(x)s(y)^{-1} &= \phi(x,y)s(yxy^{-1}), \\ f(y,x)s(yx) &= \phi(x,y)s(yxy^{-1})s(y) \\ &= \phi(x,y)f(yxy^{-1},y)s(yxy^{-1}y), \end{aligned}$$

Lemma. $\forall x, y, z \in G$,

$$
\begin{aligned}
&{}^{s(z)}\phi(x,y)\phi(x*y,z) \\
&= \phi(y,z)^{s(y*z)}\phi(x,z)\phi(x*z,y*z)^{s((x*y)*z)}\phi(y,z)^{-1}
\end{aligned}
$$

Pf. On the one hand,

$$\begin{aligned}
&(s(x) * s(y)) * s(z) \\
&\quad = [\phi(x,y)s(x*y)] * s(z) \\
&\quad = s(z)\phi(x,y)s(x*y)s(z)^{-1} \\
&\quad = {}^{s(z)}\phi(x,y)s(x*y) * s(z) \\
&\quad = {}^{s(z)}\phi(x,y)\phi(x*y,z)s((x*y)) * z)
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&(s(x) * s(z)) * (s(y) * s(z)) \\
&= [\phi(x,z)s(x*z)] * [\phi(y,z)s(y*z)] \\
&= [\phi(y,z)s(y*z)][\phi(x,z)s(x*z)][\phi(y,z)s(y*z)]^{-1} \\
&= \phi(y,z)^{s(y*z)}\phi(x,z) \\
&\quad s(x*z) * s(y*z)\phi(y,z)^{-1} \\
&= \phi(y,z)^{s(y*z)}\phi(x,z) \\
&\quad \phi(x*z,y*z)s((x*z)*(y*z))\phi(y,z)^{-1} \\
&= \phi(y,z)^{s(y*z)}\phi(x,z)\phi(x*z,y*z) \\
&\quad {}^{s((x*y)*z)}\phi(y,z)^{-1}s((x*y)) * z)
\end{aligned}$$

and we obtain the result.

Since ker π = im α, $\exists\tilde{\phi} : G \times G \times G \to N$ and $\exists\psi : G \times G \times G \to A$ is

$$^{s(z)}\tilde{\phi}(x,y)\tilde{\phi}(x*y,z)^{s((x*y)*z)}\tilde{\phi}(y,z)$$

$$= i(\psi(x,y,z))\tilde{\phi}(y,z)^{s(y*z)}\tilde{\phi}(x,z)\tilde{\phi}(x*z,y*z)$$

THM. ψ is a quandle 3-cocycle.

$$\begin{aligned} & s(w)\psi(x,y,z) + \psi(x*z,y*z,w) \\ + &\quad s((y*z)*w)\psi(x,z,w) + \psi(y,z,w) \\ = &\quad (((x*y)*z)*w)\psi(y,z,w) + \psi(x*y,z,w) \\ + &\quad s(z*w)\psi(x,y,w) + \psi((x*w,y*w,z*w). \end{aligned}$$

Pf. Consider the element in N:

$$
{}^{s(w)s(z)}\tilde{\phi}(x,y)^{s(w)}\tilde{\phi}(x*y,z)
$$
$$
{}^{s(w)s((x*y)*z)}\tilde{\phi}(y,z)
$$
$$
\tilde{\phi}((x*y)*z),w)^{s(((x*y)*z)*w)}\tilde{\phi}(y*z,w)
$$
$$
{}^{s(((x*y)*z)*w)s((y*z)*w)}\tilde{\phi}(z,w).
$$

The two ways to change that to:

$$
\tilde{\phi}(z,w)^{s(z*w)}\tilde{\phi}(y,w)\tilde{\phi}(y*w,z*w)
$$
$$
{}^{s(z*w)s(y*w)}\tilde{\phi}(x,w)
$$
$$
{}^{s((y*z)*w)}\tilde{\phi}(x*w,z*w)\tilde{\phi}((x*z)*w,(y*z)*w)
$$

are the tetrahedra move.

• There is a “new invariant” when the group action is trivial it gives the same result as the original CJKLS invariant.

• I haven’t told the whole story. Some more can be done with AG-homology. Part of Masahico’s talk.

• This seems closely related to the crossed module structure defined on the quandle.

• Example computations.