Invariants of Knots and Knotted Surfaces

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- Quandles and Their Modules

- AG cohomology

- Cocycles and Crossings

- Pregel's result

- Module Invariants

- Non-Abelian Invariants

- State-Sum Invariants

A QUANDLE is a set, Q, with a binary operation $*$ defined such that

- $a * a = a$

- $\forall a, b \in Q \; \exists! c \in Q$ s.t. $a = c * b$

- $(a * b) * c = (a * c) * (b * c)$

Examples:

(1) Groups under conjugation

(2) Λ-modules

Quandle Modules

Let X be a quandle. Let $\Omega(X)$ be the free $\mathbb{Z}$-algebra generated by $\eta_{x,y}$, $\tau_{x,y}$ for $x, y \in X$ such that $\eta_{x,y}$ is invertible for every $x, y \in X$. Define $\mathbb{Z}(X)$ to be the the quotient $\mathbb{Z}(X) = \Omega(X)/R$ where R is the subalgebra generated by

1. $$\eta_{x*y,z}\eta_{x,y} - \eta_{x*z,y*z}\eta_{x,z}$$

2. $$\eta_{x*y,z}\tau_{x,y} - \tau_{x*z,y*z}\eta_{y,z}$$

3. $$\tau_{x*y,z} - \eta_{x*z,y*z}\tau_{x,z} - \tau_{x*z,y*z}\tau_{y,z}$$

4. $$\tau_{x,x} + \eta_{x,x} - 1$$

A *repr.* of $\mathbb{Z}(X)$: Group G tog w/ $\eta_{x,y} \in$ Aut(G), $\tau_{x,y} \in$ End(G) ie. Alg homom: $\mathbb{Z}(X) \to$ End(G) Then G is a $\mathbb{Z}(X)$-module, or *a quandle module*.

Ex.

(1) Any G_X-module M, where

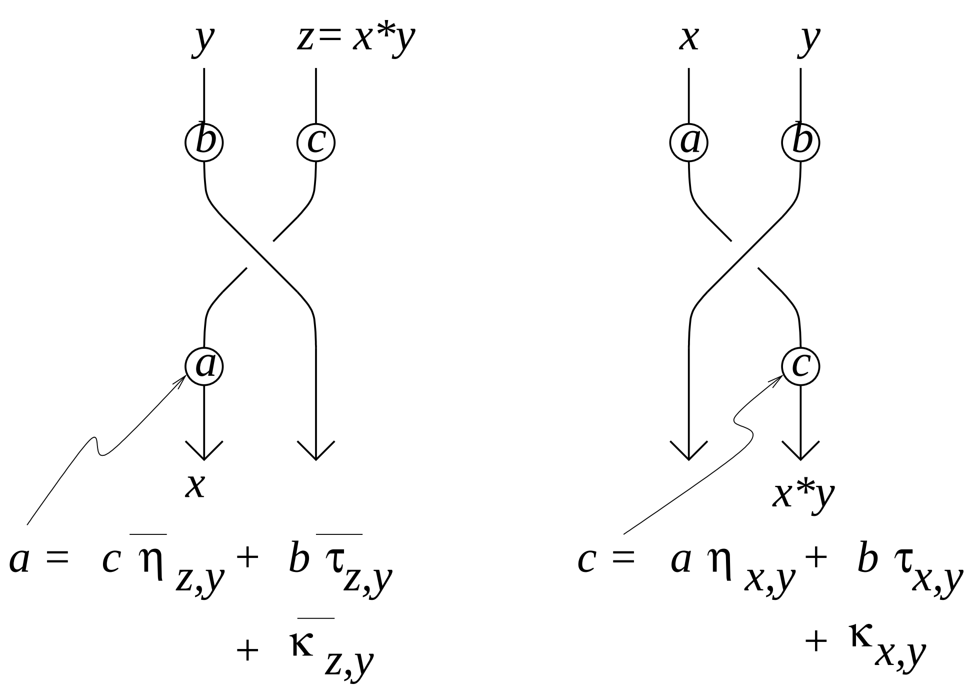
$G_X = \langle x \in X \mid x * y = yxy^{-1} \rangle$

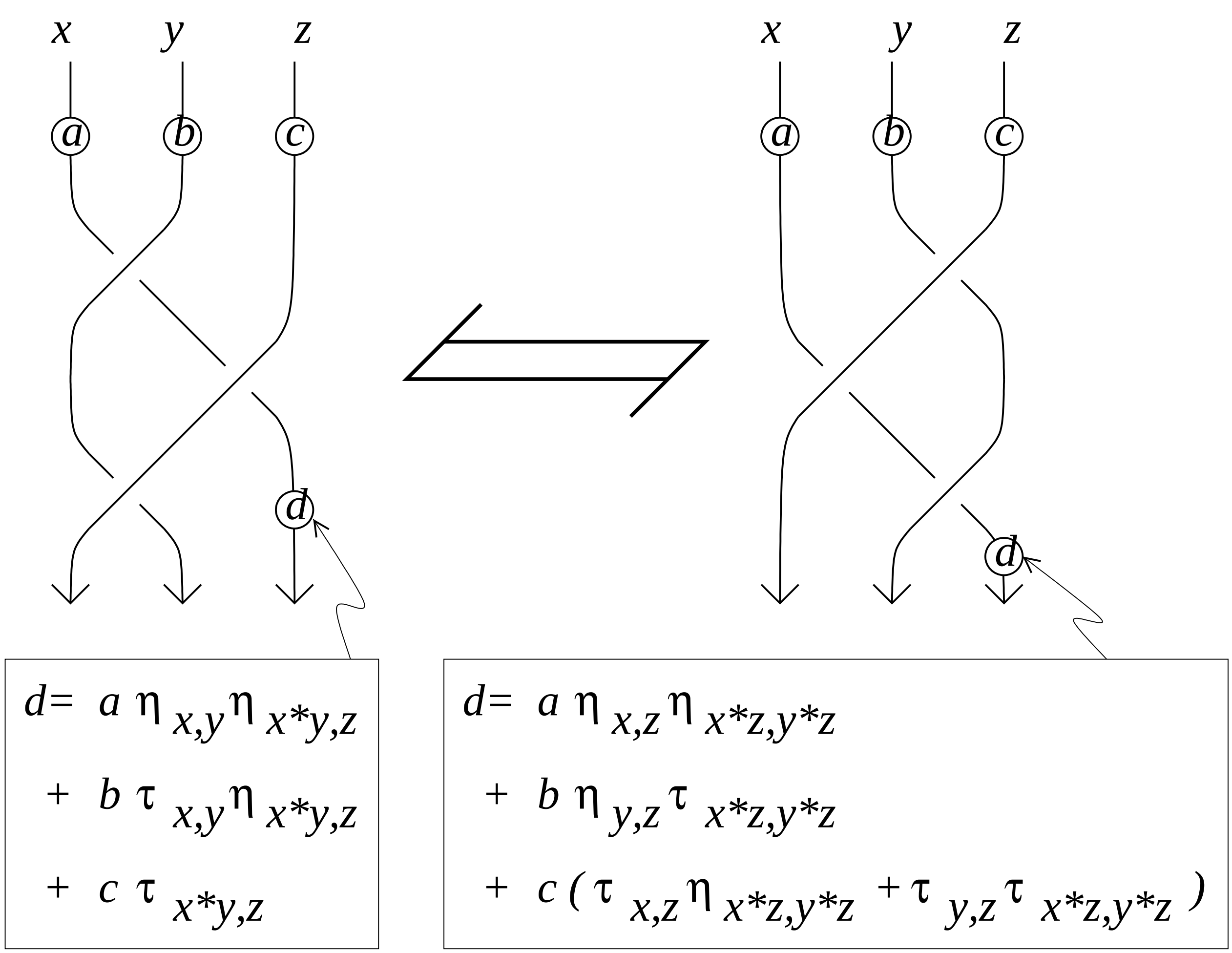
$\eta_{x,y}(a) = ya$ and $\tau_{x,y}(b) = (1 - x * y)(b)$

(2) Any Λ-module M,

$\eta_{x,y}(a) = ta$

$\tau_{x,y}(b) = (1 - t)(b)$




x y z
a b c
d
$d = a\,\eta_{x,y}\,\eta_{x*y,z}$
$+\ b\,\tau_{x,y}\,\eta_{x*y,z}$
$+\ c\,\tau_{x*y,z}$
x y z
a b c
d
$d = a\,\eta_{x,z}\,\eta_{x*z,y*z}$
$+\ b\,\eta_{y,z}\,\tau_{x*z,y*z}$
$+\ c\,(\tau_{x,z}\,\eta_{x*z,y*z} + \tau_{y,z}\,\tau_{x*z,y*z})$

$C_n(X) = \mathbb{Z}(X)X^n$ is the free right $\mathbb{Z}(X)$-module with basis X^n. $\partial = \partial_n : C_{n+1}(X) \to C_n(X)$

$$\partial(x_1, \dots, x_{n+1}) = \sum_{i=2}^{n+1}$$

$$(-1)^i \eta_{[x_1,\dots,\widehat{x_i},\dots,x_{n+1}],[x_i,\dots,x_{n+1}]}$$

$$(x_1, \dots, \widehat{x_i}, \dots, x_{n+1})$$

$$-\sum_{i=2}^{n+1} (-1)^i (x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_{n+1})$$

$$+(-1)^i \tau_{[x_1,x_3,\dots,x_{n+1}],[x_2,x_3,\dots,x_{n+1}]}(x_2, \dots, x_{n+1})$$

where

$$[x_1, x_2, \dots, x_n] = ((\cdots(x_1 * x_2) * x_3) * \cdots) * x_n.$$

2-cocycle condition

$$\eta_{x*y,z}(\kappa_{x,y}) + \kappa_{x*y,z}$$

$$= \eta_{x*z,y*z}(\kappa_{x,z}) + \tau_{x*z,y*z}(\kappa_{y,z}) + \kappa_{x*z,y*z},$$

for any $x, y, z \in X$.

$$\kappa_{x,x} = 0$$

3-cocycle condition

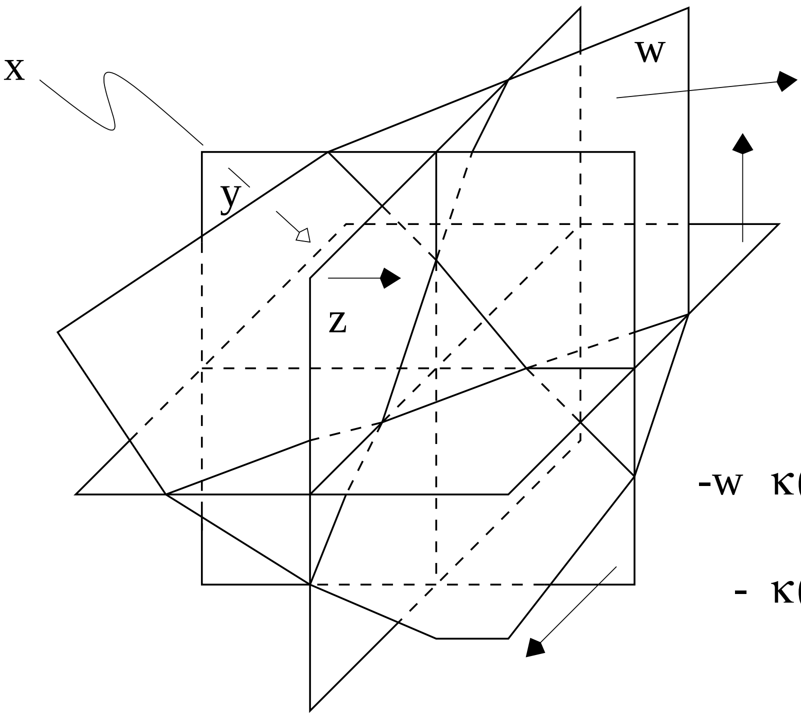
$$w\kappa_{x,y,z} + \kappa_{x*z,y*z,w} + ((y*z)*w)\kappa_{x,z,w} + \kappa_{y,z,w}$$

$$= (((x*y)*z)*w)\kappa_{y,z,w} + \kappa_{x*y,z,w}$$

$$+(z*w)\kappa_{x,y,w} + \kappa_{x*w,y*w,z*w}$$

And

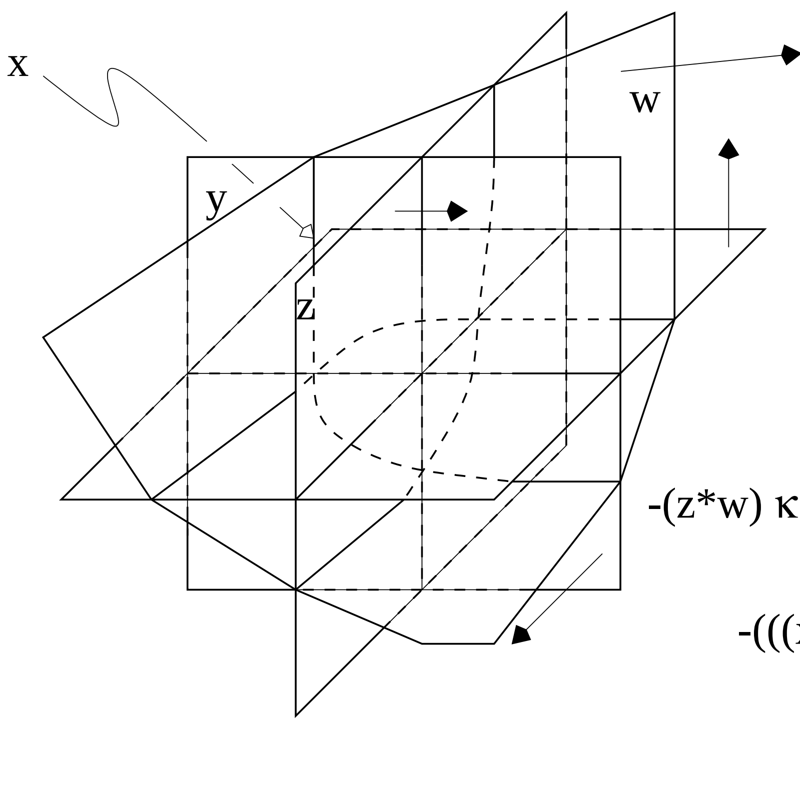
$$\kappa(x,x,y) = \kappa(x,y,y) = id.$$



- κ(y,z,w)

-w κ(x,y,z) + κ(x*y,z,w)

- κ(x*z,y*z,w)



+ κ(x*y,z,w)

-(z*w) κ(x,y,w) - κ(x*w,y*w,z*w)

-(((x*y)*z)*w) κ(y,z,w)

Letting $\eta_{x,y} = 1$ and $\tau_{x,y} = 0$ we get the homology theory given by FRS, and revised in CJKLS. In particular, there is a state-sum invariant defined for classical knots and knotted surfaces.

Classical Case: A qaundle coloring (by the quandle X) of a classical knot is given. A cocycle in $Z^2_{\mathrm{Q}}(X; A)$ is chosen. Then the cocycle is evaluated at the incoming colors to a crossing. The product of all cocycle values at the crossings, and then a sum is taken over all colorings. The quandle cocycle conditions give that this is an invariant.

Graña and Preygel:

Computations for knots in the table by Thistlethwaite, up to 16 crossings, total 1701935 knots:

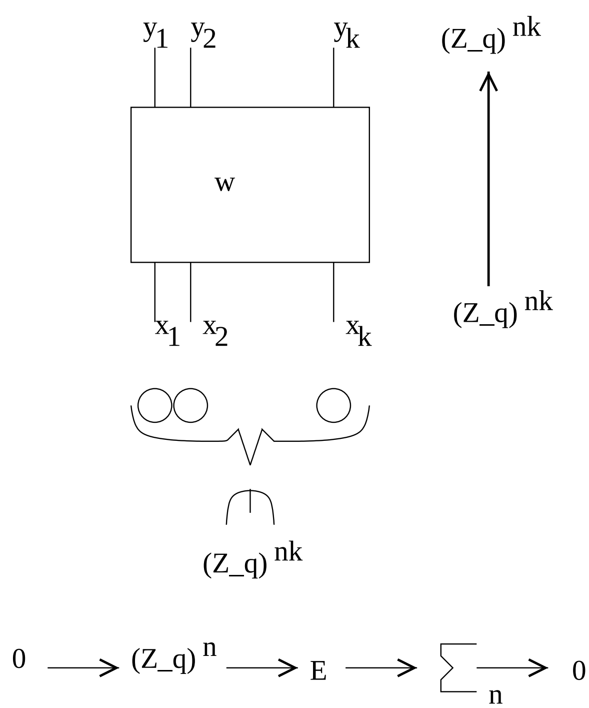
Quandle	# Inv. values	# ! inv. values
Tet	11	1
Cube	763	179
Dodec	16114	5887
Icos	5934	1713
Trans in S_4	40	0
Trans in S_5	57	0
(2)(2) in S_5	410	46
(2)(3) in S_5	4793	1896
(4) in S_5	8903	4259
Trans in S_6	52	3
totals	694945	514160

Module invariants

- w is a k-braid word with closure $\widehat{w}$.

- $X \subset \text{Conj}(\Sigma_n)$, $E = (\mathbb{Z}_q)^n \rtimes \Sigma_n$, and $\tilde{X} = \pi^{-1}(X)$.

- α is a dyn cocycle in the sense of AG. so that $\tilde{X} = (\mathbb{Z}_q)^n \times_\alpha X$. $\alpha_{x,y}(a,b) = y(a) + (1 - x * y)(b)$



THM.*Let* $L = \widehat{w}$, *Col*$_X(L)$ *be the set of colorings of* L *by a quandle* X. *For* $\mathcal{C} \in$ *Col*$_X(L)$, *let* $\vec{x}$ *be the color vector of bottom strings of* w *that is the restriction of* $\mathcal{C}$. *Then the family*

$$\tilde{\Phi}(X, \alpha\,; L) = \{(\mathbb{Z}_q)^{nk}/\mathrm{Im}(M(w, \vec{x}) - I)\}_{\mathcal{C} \in Col_X(L)}$$

of isomorphism classes of modules presented by the maps $(M(w, \vec{x}) - I)$, *is a link invariant.*

Preliminary Compuations: Non-trivial values exist, and when given coloring is trivial, we are picking up the Det. In fact, Masahico has run Maple a program for through 8 crossings, but this has not been checked with Mathematica, because the latter does not have smith normal forms canned.

Non-Abelian Cocycles $\beta : X \times X \rightarrow H$ (where H is a group but not-nec. Abel.) is a *rack 2-cocycle*

$$\beta(x_1, x_2)\beta(x_1 * x_2, x_3) = \beta(x_1, x_3)\beta(x_1 * x_3, x_2 * x_3)$$

(Concept from Andruskiewitsch and Graña)

We can define state-sum invariant via non-abelian cocycles. We have some preliminary computations.

THM (CEGS): There is a State-sum invariant for knotted surfaces using such cocycles. Cohomologous cocycles yield the same values.

3-cocycle condition

$$w\kappa_{x,y,z} + \kappa_{x*z,y*z,w} + ((y*z)*w)\kappa_{x,z,w} + \kappa_{y,z,w}$$

$$= (((x*y)*z)*w)\kappa_{y,z,w} + \kappa_{x*y,z,w}$$

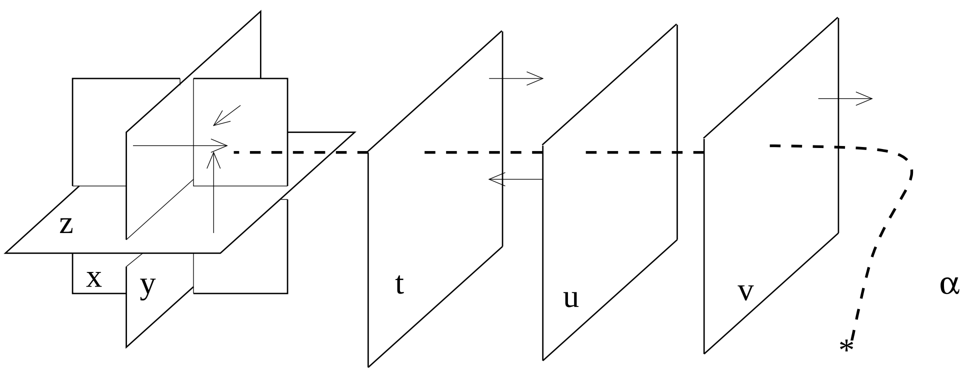
$$+ (z*w)\kappa_{x,y,w} + \kappa_{x*w,y*w,z*w}$$

And

$\kappa(x,x,y) = \kappa(x,y,y) = id.$

Sketch:(a) The quandle cocycle is evaluated at source regions of triple points, taking the product over all triple points, and summing over all colorings.

(b) In fact, we take an arc from ∞ to the target region in a KSD. Then the labels on the regions that this arc crosses act on the value of the cocycle at these evaluations.



$- \ v^{-1} \ u \ t^{-1} \ \kappa_{x,y,z}$ = Boltzmann weight at triple point

Examples We let the transpositions $(23), (13)$, and (12) act as permutations of the arguments for functions $\kappa : R_3^3 \rightarrow \mathbb{Z}_3$. Then we use a Maple program to find a function κ that satisfies the cocycle condition. Mathematica also constructs functions with values in $\mathbb{Z}$ that satisfy these conditions. We have verified with Mathematica that the functions that Maple produces work. In fact, the space of solutions is spanned by two functions, $q[1]$, and $q[2]$.

We have run computations the values of these cocycles on 2-twist-spins of all the prime knots through 8 crossings that are 3-colorable. We have found non-trivial values for these. These last computations need to be checked.

$$\kappa[0,1,0] = (2q[2], q[1], 2q[2]),$$

$$\kappa[0,1,2] = (q[1]+2q[2], 2q[1]+q[2], q[1]+q[2]),$$

$$\kappa[0,2,0] = (q[2], 2q[1]+2q[2], 2q[2]),$$

$$\kappa[0,2,1] = (q[2], 0, 2q[1]+q[2])$$

$$\kappa[1,0,1] = (2q[1], q[1]+q[2], 0)$$

$$\kappa[1,0,2] = (q[1], 0, q[1])$$

$$\kappa[1,2,0] = (2q[1], q[1]+2q[2], 0)$$

$$\kappa[1,2,1] = (q[1], 0, 0)$$

$$\kappa[2,0,1] = (2q[1], q[1], 0)$$

$$\kappa[2,0,2] = (q[1]+2q[2], 2q[1], 0)$$

$$\kappa[2,1,0] = (q[2], 0, 0)$$