Elementary Mathematics from an Advanced Viewpoint

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March 23, 2005

Chapter 1

Overview

Our goal is to become familiar with several aspects of college level mathematics. Specifically, we want to have working knowledge of linear, quandratic, exponential, logarithmic and trigonmetric phenomena. Our approach is to start from the elementary and build towards the advanced. The motto of a very famous program for high school students (The Ross Program) is, “To think deeply about elementary things.” [Check with Ross to get exact quote].

The introductory chapter involves arithemetic from an algebraic point of view. Too often, mathematics books start from review chapters that consist of things that are not well understood by the reader. The result is that students are initially confused and frustrated. On the other hand, people feel that they understand the four basic arithmetic operations even if they are not accurate with their results. But they often begin to have trouble when symbolic representations of number concepts are introduced. We view algebra as a codification of patterns that are seen among arithmetical computations. Second, mathematics, very often, is the search for patterns and overarching results that can be applied to a variety of situations. There are beautiful patterns in arithmetic that you should know about. You cannot be expected to have “number sense” if you do not have experience in doing some arithmetic. Finally, we want to introduce the complex number system which is a method of doing arithmetic in the plane. It is better to introduce this early when the realm of discourse is arithmetic, rather than later when the realm of discourse is algebra. Learning to compute with complex numbers will facilitate your making algebraic computations. Meanwhile, we can introduce the basic concepts of trigonmetry in a numeric setting.

The second chapter involves all things linear. We learn the basics of graphing lines and planes, computing their intersections, and generalizing to invisible spaces. A lot of time and effort is spent here. We want students to know every possible representation of a line in the plane, and a plane in space. We want students to recognize that simple equations represent simple geometric concepts.

The third chapter involves a discussion of all things quadratic. In the plane quadratic phenomena include the basic shapes of circles, ellipses, hyperbolas and parabolas. In space, they are certain surfaces: Spheres, ellipsoids, hyperboliods, satellite dishes, and so forth. We discuss quadratic phenomena in which more than 3-variables are present. The main algebraic lessons that are learned are: (1) the method of completing the square [this is informed by computations from chapter 1], (2) the matrix representation of quadratic forms [this is informed by lessons learned in chapter 2].

The fourth chapter, deals directly with polynomial and rational expressions in two variables for which one variable can be expressed unambiguously in terms of the other. The graphical representations are curves in which no vertical line intersects the graph more than once. this concept is the basic function concept. But we are not working with the general concept of function, we are still maintaining a focus on specific examples. Our main focus is to develop the notions of factoring, and determining large scale qualitative analyses of graphs with minimal computation. Furthermore, we point our specific analogies between the arithemetical properties of polynomials and numbers.

The fifth chapter involves the fifth arithmetical operation. Beyond addition, subtraction, multiplication, and addition, there is an operation called exponentiation. Studies of exponential functions are extremely gratifying because they point to methods of securing your future: making house payments and planning for retirement. Moreover, exponential growth and decay are among the most studied and well understood phenomena after the linear and quadratic. Most importantly, we relate the exponential and trigonmetric phenomena within this chapter via the use of complex numbers. Thus we recapituate ideas that were introduced in chapter 1.

The most important aspect of this development is the uniified and integrated approach that exercises play in the book. Each set of exercises develops a basic concept. Mastery is required before the student continues. Sections in the text develop first the problem solving skill, then the text develops the conceptual basis. We do what we can to foreshadow upcoming discussions and to indicate applications and how these concepts relate to the broader body of mathematics. In particular, when an analysis requires a concept from calculus then we develop that concept within the context of the chapter. Futhermore, whenever possible we indicate ideas of proof. We expect students to learn these proofs and they are tested within the course.

1. Arithmetic
   (a) The sequence of square numbers
   (b) mental computations for multiplication [difference of squares]
   (c) Adding numbers in patterns [Introduction to Induction]
   (d) Approximation of square roots [Basic 2 digit approximations]
   (e) Introducing the square root of -1.
   (f) Ways to measure in the plane [distance and angle]
   (g) The four basic arithmetic operations [field axioms]
2. Linear Phenomena
   (a) x, y conventions.
   (b) See details Below
3. Quadratic Phenomena
4. Absolute Value Functions
5. Polynomial Functions
6. Rational Functions
7. Exponential and Log Functions

8. Trigonometric Functions

9. Radical Expressions

Chapter 2

Linear Expressions

In general, suppose that $x_1, x_2, \ldots, x_n$ are VARIABLES — quantities whose values we don't know or which may change. Meanwhile, suppose that $A_1, A_2, \ldots, A_n$, and R are CONSTANTS— quantities whose value we do know. A LINEAR EQUATION is an equation of the form

$$A_1x_1 + A_2x_2 + \cdots A_nx_n = R.$$

When the number, n, of variables is small, then we use x, y, z *etc.* for our variables and A, B, C for our constants. The letter R stands for the right-hand-side of the equation. Its role in the theory will become more clear within context.

Examples.

1. The equation, $3x + 2y = 6$, is a linear equation. The set of solutions to this equation is a line in the (x, y)-plane.

2. The equation $2x = 4$ is a linear equation in one variable, x. Its solution is obtained as follows:
$$\begin{aligned} 2x &= 4 \\ \frac{2x}{2} &= \frac{4}{2} \\ x &= 2 \end{aligned}$$

3. The equation $2x - 4y + 3z = 5$ is a linear equation in 3-variables, x, y, z. The equation represents a plane in 3-dimensional (x, y, z)-space.

4. The equation $5x + 2y - z + w = 14$ is a linear equation in 4-variables, x, y, z, w. [The alphabet ran out of letters before we ran out of equations. Mathematicians tend to let w be the last of 4-variables rather than first.] The equation has geometric meaning as a flat 3-dimensional subspace of 4-dimensional space. The 4-D space in question is the space of all ordered 4-tuples (x, y, z, w) in which the variables can take values among the real numbers. We will return to this example later.

5. When more than four variables are present we start to subscript the variables. So the equation $6x_1 - 4x_2 - 3x_3 + 2x_4 - 7x_5 = 2$ is a linear equation in 5-variables.

Each of these examples is given in GENERAL FORM. There are advatages to working with the general form of a linear equation. One goal that we have is to be able to switch equations among their vairous forms. Another is to interpret the coefficients A, B, C *etc.* in the general form.

2.1 Equations of Lines

One of the most basic notions of planar geometry is the following:
Fundamental Geometric Axiom: Any two points determine a line.

We begin with pairs of points in the plane, and we compute quantities of the line that is so determined.

2.1.1 Slope of a Line

The SLOPE of the line that is determined by the points (x_1, y_1) and (x_2, y_2) is the quantity:

$$M = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}.$$

The symbol Δ is called a delta. It means the change in the given variable. So Δy means the change in y, or $y_2 - y_1$.

Figure 2.1 indicates the definition of slope.

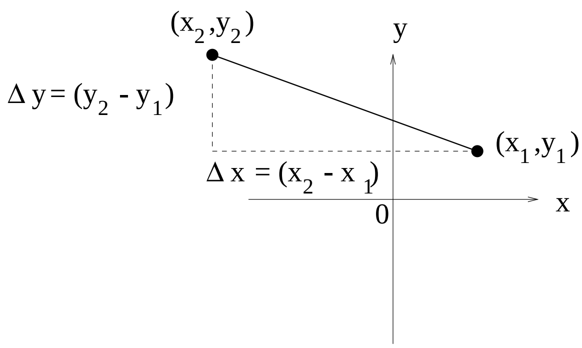

Figure 2.1: The Definition of slope

Example. Consider the points $(1, 3) = (x_1, y_1)$ and $(2, 5) = (x_2, y_2)$. The change in y is given as

$$\Delta y = y_2 - y_1 = 5 - 3 = 2.$$

The change in x is given as

$$\Delta x = x_2 - x_1 = 2 - 1 = 1.$$

The slope of the line between the points is given as

$$M = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 3}{2 - 1} = 2.$$

Notational conventions. Always leave the slope as a reduced fraction. However, if the slope is an improper fraction, such as $5/2$, *do not* convert to proper form $2\frac{1}{2}$. When graphing, use the denominator of the slope, as the units along the x-axis and the numerator as the units along the y-axis unless you are graphing two lines in the same coordinate system.

If x does not change, so that $\Delta x = 0$, then there is a value of 0 in the denominator of the slope. In this case, we say the line has "no slope." If $\Delta y = 0$, then y does not change, so the slope is 0 in this case. When y does not change the line is a constant function, and the slope of a constant is 0. HAVING "NO SLOPE" IS DIFFERENT THAN HAVING SLOPE, $M = 0$!

Exercises

For each pair of points given below: graph the points, and compute the slope of the line that passes between the points.

1. $(x_1, y_1) = (1, 0),\ (x_2, y_2) = (0, 1)$
2. $(x_1, y_1) = (1, 0),\ (x_2, y_2) = (1, 1)$
3. $(x_1, y_1) = (1, 0),\ (x_2, y_2) = (2, 1)$
4. $(x_1, y_1) = (-1, 0),\ (x_2, y_2) = (0, 1)$
5. $(x_1, y_1) = (3, 1),\ (x_2, y_2) = (2, 3)$
6. $(x_1, y_1) = (-1, 2),\ (x_2, y_2) = (0, -1)$
7. $(x_1, y_1) = (1, 3),\ (x_2, y_2) = (2, -4)$
8. $(x_1, y_1) = (-2, 0),\ (x_2, y_2) = (0, -5)$
9. $(x_1, y_1) = (1, 1),\ (x_2, y_2) = (3, 2)$
10. $(x_1, y_1) = (-3, -8),\ (x_2, y_2) = (3, 4)$
11. $(x_1, y_1) = (2, 1),\ (x_2, y_2) = (-1, 4)$
12. $(x_1, y_1) = (4, -3),\ (x_2, y_2) = (-1, -1)$
13. $(x_1, y_1) = (5, 12),\ (x_2, y_2) = (-4, 3)$
14. $(x_1, y_1) = (0, 0),\ (x_2, y_2) = (3, 1)$
15. $(x_1, y_1) = (0, 0),\ (x_2, y_2) = (6, 7)$
16. $(x_1, y_1) = (-1, 0),\ (x_2, y_2) = (0, 0)$
17. $(x_1, y_1) = (-1, -10),\ (x_2, y_2) = (1, 4)$
18. $(x_1, y_1) = (0, 0),\ (x_2, y_2) = (0, 5)$
19. $(x_1, y_1) = (2, 1),\ (x_2, y_2) = (4, 1)$

20. $$(x_1, y_1) = (-7, 4),\ (x_2, y_2) = (3, 1)$$

21. $$(x_1, y_1) = (-2, 2),\ (x_2, y_2) = (3, 10)$$

22. $$(x_1, y_1) = (-3, 8),\ (x_2, y_2) = (0, 1)$$

23. $$(x_1, y_1) = (-5, 0),\ (x_2, y_2) = (0, 6)$$

24. $$(x_1, y_1) = (-14, 0),\ (x_2, y_2) = (0, 10)$$

25. $$(x_1, y_1) = (-6, 2),\ (x_2, y_2) = (-3, 4)$$

26. $$(x_1, y_1) = (-1, -1),\ (x_2, y_2) = (3, 5)$$

27. $$(x_1, y_1) = (-15, -4),\ (x_2, y_2) = (0, 10)$$

28. $$(x_1, y_1) = (1, -3),\ (x_2, y_2) = (2, -1)$$

29. $$(x_1, y_1) = (1, -2),\ (x_2, y_2) = (2, -1)$$

30. $$(x_1, y_1) = (-1, -20),\ (x_2, y_2) = (3, -1)$$

2.1.2 Angle determined by a line

Extended Example. Suppose that $(x_1, y_1) = (0, 0)$, and that $(x_2, y_2) = (3, 4)$. Then we compute the slope as

$$M = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{3 - 0} = 4/3.$$

The DISTANCE between the two points is

$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(3-0)^2 + (4-0)^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

The angle Θ (the symbol Θ is pronouned theta) in the Figure 2.2 is the angle that the line makes with the x-axis. In this example, we have put one of the points at the origin tomake the situation easier to see. In general though, the line will intersect the x-axis at some other point. Associated to this angle are three ratios.

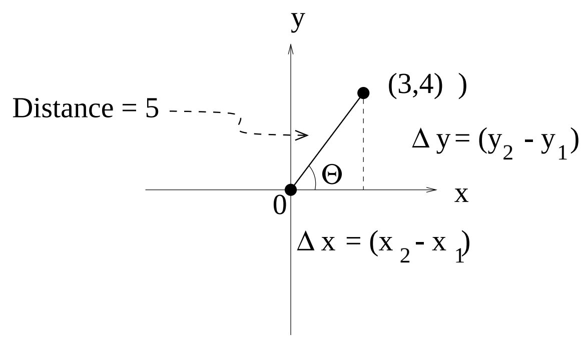

Figure 2.2: The Angle defined by a line

The sine of the angle is written as, $\sin\Theta$. We compute it as

$$\sin\Theta = \frac{\Delta y}{D} = 4/5.$$

In general, the SINE OF AN ANGLE is the ratio of the side opposite the angle and the length of the hypotenuse.

The cosine of the angle is written as, $\cos\Theta$. We compute it as

$$\cos\Theta = \frac{\Delta x}{D} = 3/5.$$

In general, the COSINE OF AN ANGLE is the ratio of the side adjacent to the angle and the length of the hypotenuse.

The tangent of the angle is written as, $\tan\Theta$. We compute it as

$$\tan\Theta = \frac{\Delta y}{\Delta x} = \frac{4}{3}.$$

In general, the TANGENT OF AN ANGLE is the ratio of the side opposite the angle and side adjacent to the angle. THE TANGENT OF AN ANGLE IS ANOTHER NAME FOR THE SLOPE.

When the line has no slope (so there is no change in x, *i.e.* $\Delta x = 0$), then we say that the angle is vertical, and by definition $\Theta = 90°$.

Exercises

For each pair of points given above, compute the distance between the points. LEAVE YOUR ANSWER AS A RADICAL! That is if the distance is $\sqrt{5}$, leave this as your answer. Compute the sine, cosine, and tangent of the angle that the line makes with the positive x-axis. To check your answer, change the calculator mode to being in Degrees, and find the degree measurement of the resulting angle. (I'll show you how with the calculator). The angles that you get should all be the same for the same problem.

For example, Suppose that we computed $\sin\Theta = 3/5$, and $\cos\Theta = 4/5$, but got that $\tan\Theta = 4/3$. On the cacluator, we would have found that $\Theta = 37°$ in the first and second cases but $\Theta = 53°$ in the last case. If the numbers came from the same problem, then there was a mistake.

2.1.3 Point-Slope Form

The POINT-SLOPE FORM of the equation of a line is an equation in the form $(y-y_1) = M(x-x_1)$ where M denotes the slope, and (x_1, y_1) denotes a point on the line. For example suppoe that $(x_1, y_1) = (8, 7)$ and $(x_2, y_2) = (1, 4)$. Then we compute that

$$M = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-7}{1-8} = \frac{-3}{-7} = 3/7.$$

The equation of the line that passes between these points is

$$(y-7) = \frac{3}{7}(x-8).$$

This is the equation in point-slope form. We use it when we know or can find the slope, and we know a point on the line. There is no reason, yet to simplify this equation.

Exercises.

For each of the 30 pairs of points above, compute the equation of the line in point-slope form. In case the line has no slope the equation will be of the form $x = x_1$. For example, the line through $(3, 1)$ and $(3, 7)$ has no slope (Check this!). The line is vertical, and its equation is $x = 3$.

2.1.4 Slope-Intercept Form

The SLOPE-INTERCEPT FORM of the equation of the line is the equation

$$y = Mx + B$$

where $(0, B)$ is the y-intercept and M is the slope. If the equation is given in point-slope form it is easy to convert to slope intercept form. Taking the equation $(y - 7) = \frac{3}{7}(x - 8)$ we indicate how to convert it to slope-intercept form:

$$\begin{aligned}
(y - 7) &= \frac{3}{7}(x - 8) \\
y - 7 &= \frac{3}{7}x - (\frac{3}{7})8 \\
y - 7 &= \frac{3}{7}x - (\frac{3}{7})\frac{8}{1} \\
y - 7 &= \frac{3}{7}x - \frac{3 \cdot 8}{7 \cdot 1} \\
y - 7 + 7 &= \frac{3}{7}x - \frac{24}{7} + 7 \\
y &= \frac{3}{7}x - \frac{24}{7} + \frac{49}{7} \\
y &= \frac{3}{7}x + \frac{25}{7}
\end{aligned}$$

The steps in this process are, in order, (1) Distribute the 3/7th across the parentheses on the right, (2) convert 8 to a fraction, (3) multiply fractions, (4) complete the multiplications and add 7 to both sides of the equation, (5) the quantity $-7 + 7 = 0$ and convert 7 to 49/7 so that we can add fractions at the next step, and finally (6) add the fractions.

We learn from this that the y-intercept is 25/7 or $3\frac{4}{7}$.

Exercises.

Convert the equations above that you have in point-slope form to equations in slope-intercept form. Check your answers by examining your graphs, and see if the point $(0, B)$ is a solution to the original equation.

Computer Exercise.

Plot the pair of points on MATHEMATICA, plot the resulting line (in point slope form) and show the two plots on the same window. If your points are not on your line there is a mistake. The Mathematica commands for this are indicated for an example here:

```
ListPlot[{{8, 7}, {1, 4}},
  PlotStyle -> PointSize[.02]]

Plot[3/7 x + 25/7, {x, -14, 14}]

Show[%, %%]
```

2.1.5 Slope-Intercept Form

The SLOPE-INTERCEPT FORM of an equation allows for an easy method of graphing a function. Figure slopeint indicates the steps.

Exersices

For each of the equations below, graph the line by (1) indicating the y-intercept, (2) using the slope to find a second point on the line, and (3) determine algebraically the value of the x-intercept.

1. $y = 2x - 4$
2. $y = \frac{2}{3}x + 3$
3. $y = \frac{-2}{5}x - 4$
4. $y = \frac{5}{7}x + 3$
5. $y = 5x - \frac{1}{2}$
6. $y = \frac{3}{7}x + \frac{4}{9}$
7. $y = -2x - 4$

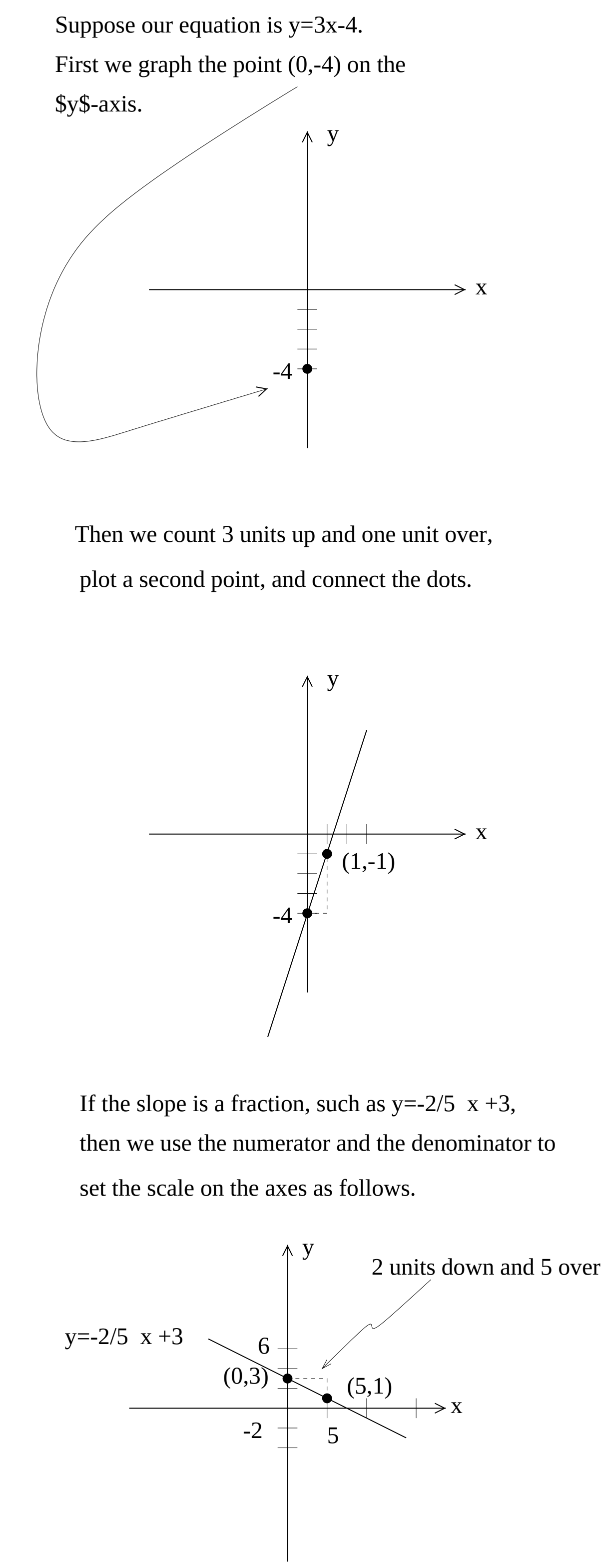

Figure 2.3: The slope-intercept form of a line

8. $y = x + 3$

9. $y = x$

10. $y = \frac{5}{x} - 30$

2.1.6 The General Form

THE GENERAL FORM of the equation of a line is the equation

$$Ax + By = R$$

where A, B, and R are constants. For example, the equation $3x + 2y = 7$ is an equation in general form.

For this line, $3x + 2y = 7$, we can put it in point slope form as follows:

$$\begin{aligned}
3x + 2y &= 7 \\
-3x + 3x + 2y &= -3x + 7 \\
0 + 2y &= -3x + 7 \\
2y &= -3x + 7 \\
\frac{1}{2}2y &= \frac{1}{2}(-3x + 7) \\
y &= \frac{-3}{2}x + \frac{7}{2}
\end{aligned}$$

So the slope is $-3/2$ and the y-intercept is 72.

Theorem. *If $B \neq 0$, then the slope of the line $Ax + By = R$ is $-A/B$. The y-intercept is R/B. The x-intercept is R/A.*

Observation. If we divide the y-intercept by the x-intercept, and change the sign of the result we obtain the slope. For example,

$$\frac{\left(\frac{R}{B}\right)}{\left(\frac{R}{A}\right)} = \left(\frac{R}{B}\right)\left(\frac{A}{R}\right) = \frac{A}{B} = -M.$$

There are many ways to see this. The easiest is to compute the slope from the points $(x_1, y_1) = (\frac{R}{A}, 0)$ and $(x_2, y_2) = (0, \frac{R}{B})$. During this calculation, you will encounter complex fractions and a negative sign *no matter how you do the arithmetic.* It is important to learn to cope with these arithmetical difficulties.

Exercises

. For each of the following lines compute the slope, the y-intercept, the x-intecept, and sketch the graph of the line.

1. $2x - y = 5$
2. $x - 2y = 5$
3. $5x - 2y = 1$
4. $x - 5y = 2$
5. $2x - 5y = 1$
6. $-2x + 5y = 1$
7. $-2x - 5y = 1$
8. $x - 5y = -2$
9. $x - 5y = 2$
10. $x - y = 5$
11. $x + y = 5$
12. $x - y = 2$
13. $-x + y = 2$
14. $x + y = 2$
15. $x + 3y = 7$

Our goal for such problems should be to recognize whether the slope is positive or negative, and to determine the signs of the x and y-intercepts.

2.1.7 Perpendiculars

When a line $Ax+By = R$ is given we can interpret the pair (A, B) as a vector (arrow) that is perpendicular to the line. Starting from any point on the line (for example the x-intercept). Draw an arrow with x-displacement A and y displacement B from that point. Notice that this arrow is perpendicular to the line. [Later I will draw a picture to indicate this].

Exercises

Draw the perpendiculars for each of the exercises above.

2.1.8 The Intercept-Intercept Form.

The INTERCEPT-INTERCEPT form of a line is the form:

$$\frac{x}{a} + \frac{y}{b} = 1$$

This differs from the general form in that (1) THE RIGHT-HAND-SIDE IS 1. (2) The coefficients $\frac{1}{a}$ and $\frac{1}{b}$ of x and y, repectively, are expressed as recipricol fractions.

Examples

1. The line $\frac{x}{2} + \frac{y}{3} = 1$, has x-intercept $(2, 0)$ and y-intercept $(0, 3)$.
2. The line $4x - 7y = 1$, has x-intercept $(\frac{1}{4}, 0)$ and y-intercept $(0, \frac{-1}{7})$.
3. The line $\frac{-5x}{2} + \frac{2y}{3} = 1$, has x-intercept $(\frac{-2}{5}, 0)$ and y-intercept $(0, \frac{3}{2})$.
4. The line $3 \cdot \frac{x}{2} + 20 \cdot \frac{y}{3} = 1$, has x-intercept $(\frac{2}{3}, 0)$ and y-intercept $(0, \frac{3}{20})$.

Exercises

For each of the following pairs of points write the equation of the line in intercept-intercept form. Sketch the graph of the line.

1. $$(x_1, y_1) = (120, 0),\ (x_2, y_2) = (0, 40)$$
2. $$(x_1, y_1) = (0, 0),\ (x_2, y_2) = (0, 5)$$

3. $(x_1, y_1) = (2, 0),\ (x_2, y_2) = (0, 1)$

4. $(x_1, y_1) = (-7, 0),\ (x_2, y_2) = (0, 3)$

5. $(x_1, y_1) = (-2, 0),\ (x_2, y_2) = (0, 3)$

6. $(x_1, y_1) = (3, 0),\ (x_2, y_2) = (0, 10)$

7. $(x_1, y_1) = (-5, 0),\ (x_2, y_2) = (0, 6)$

8. $(x_1, y_1) = (14, 0),\ (x_2, y_2) = (0, 10)$

9. $(x_1, y_1) = (6, 0),\ (x_2, y_2) = (0, 4)$

10. $(x_1, y_1) = (-1, 0),\ (x_2, y_2) = (0, 5)$

11. $(x_1, y_1) = (-15, 0),\ (x_2, y_2) = (0, 10)$

12. $(x_1, y_1) = (-3, 0),\ (x_2, y_2) = (0, 2)$

13. $(x_1, y_1) = (1, 0),\ (x_2, y_2) = (0, -1)$

14. $(x_1, y_1) = (-1, 0),\ (x_2, y_2) = (0, -1)$

2.1.9 Temperature Conversion

It is well known that the Farenheit and Celcius scales of temperatures are linearly related with $0°C$ corresponding to $32°F$. Meanwhile $100°C$ corresponds to $212°F$. Thus on a coordinate system in which Farenheit is along the horizontal axis, and Celcius is along the vertical the points $(32, 0)$ and $(212, 100)$ are points on the graph.

Exercises

1. Write Celcius temperature as a function of Farenheit in slope-intercept form.

2. Using your answer from the first part, determine the Celcius temperature for each of the following Farenheit temperatures:

(a) $45°F$

(b) $57°F$

(c) $68°F$

(d) $80°F$

(e) $95°F$

(f) $98.6°F$

(g) $-10°F$

3. At what temperature is the Celcius temperature equal to the Farenheit Temperature?

2.1.10 Parametric Form

When we specify lines in 3 or more dimensions (and we will need to do this when we consider the simultaneous solutions to many linear equations in many variables), we cannot rely on slopes to determine the line. Instead, we find a vector (arrow) that points in the direction of the line. Given two points (x_1, y_1) and (x_2, y_2) such an arrow could be chosen to be $(\Delta x, \Delta y) = (x_2 - x_1, y_2 - y_1)$. Then the line is the union of the two rays that start at (x_1, y_1) and follow parallel and anti-parallel to the given arrow.

The PARAMETRIC FORM OF A LINE BETWEEN THE POINTS (x_1, y_1) AND (x_2, y_2) is given by

$$\ell(t) = (x_1, y_1) + t(\Delta x, \Delta y).$$

It would be more precise to say that

$$\{\ell(t) = (x_1, y_1) + t(\Delta x, \Delta y) : \quad t \in \mathbb{R}\}$$

is the same as the point set of the line. The expression in curly braces is read, "the set of all points of the form ell of t equals to ex sub one why sub one plus tee times delta x delta y, as tee ranges over all real numbers."

Example. Consider the pair of points $(x_1, y_1) = (8, 7)$ and $(x_2, y_2) = (1, 4)$. We determine the direction vector

$$(\Delta x, \Delta y) \quad = \quad (x_2 - x_1, y_2 - y_1)$$

$$
\begin{aligned}
&= (1-8, 4-7) \\
&= (-7, -3)
\end{aligned}
$$

The line in parametric form is

$$\ell(t) = (8,7) + t(-7,-3).$$

Let us tabulate a set of values.

t	$\ell(t)$
0	$(8,7)$
0.5	$(4.5, 5.5)$
-1	$(15, 10)$
0.75	$(2.75, 4.75)$
1	$(1,4)$
2	$(-6,1)$
-2	$(22, 13)$
$8/7$	$(0, 25/7)$
$7/3$	$(-25/3, 0)$

Exercise

Plot the points in the table above on a single graph. On the same graph sketch the line
$y = \frac{3}{7}x + \frac{25}{7}$. (The points and the line came from an example above). Observe that all the points given in the table lie on the line. Thus the line is the set of all these points for ALL real numbers t.

Exercise

For each of the original 30 points above, write
the equation of the line in parametric form.

2.1.11 Converting parametric form to Standard form

Example. Consider the line between the points $(6, 3)$ and $(-2, 7)$.

The parametric form of the line is

$$\ell(t) = (6, 3) + t(-8, 4).$$

We can write this equation as

$$(x(t), y(t)) = (6 - 8t, 3 + 4t),$$

or as a pair of equations

$$x = 6 - 8t$$

and

$$y = 3 + 4t.$$

We solve for t in the first equation as follows:

$$x + 8t = 6,$$

$$8t = 6 - x,$$

$$t = \frac{3}{4} - \frac{1}{8}x$$

Now we substitute this value of t into the second equation

$$y = 3 + 4(\frac{3}{4} - \frac{1}{8}x)$$

Upon simplification we get.

$$y = 6 - \frac{1}{2}x$$

Thus the slope is $\frac{1}{2}$ and the y-intercept is 6.

Exercises

Convert your parametric equations to slope-intercept form (where possible), and see if these equations agree with those you found originally.

2.1.12 Intersection Between Lines

. Given a pair of lines in the plane, we expect them to intersect, and indeed they will intersect at a point unless they are parallel. There are several methods to determine the x and y coordinates of the point of intersection. We will explore two of these.

Example. Consider the pair of lines

$$\begin{aligned} 2x - 3y &= 5 \\ 4x + 5y &= 6 \end{aligned}$$

In the first equation, we solve for x in terms of y :

$$\begin{aligned} 2x - 3y &= 5 \\ 2x &= 5 + 3y \\ x &= \frac{(5+3y)}{2} \end{aligned}$$

Next we take this value, and substitute this value for x into the second equation.

$$\begin{aligned} 4x + 5y &= 6 \\ 4 \cdot \frac{(5+3y)}{2} + 5y &= 6 \\ 2 \cdot (5 + 3y) + 5y &= 6 \\ 10 + 6y + 5y &= 6 \\ 10 + 11y &= 6 \\ 11y &= 6 - 10 \\ y &= -4/11 \end{aligned}$$

Finally, we substitute this value of y back into the equation $x = \frac{(5+3y)}{2}$.

$$\begin{aligned}
x &= \frac{(5+3y)}{2} \\
x &= \frac{(5+3\cdot(-4/11))}{2} \\
x &= \frac{(5+-12/11)}{2} \\
x &= \frac{1}{2}(5+-12/11) \\
x &= \frac{1}{2}\cdot\frac{(55+-12)}{11} \\
x &= \frac{1}{2}\cdot\frac{43}{11} \\
x &= \frac{43}{22}
\end{aligned}$$

Thus the point at which these lines intersect is the point $(43/22, -4/11)$.

Exercises.

For each of the following pairs of equations determine the point of intersection.

1.

$$\begin{aligned}
x - y &= 5 \\
x + y &= 6
\end{aligned}$$

2.

$$\begin{aligned}
2x + 3y &= 1 \\
4x + 5y &= 1
\end{aligned}$$

3.

$$\begin{aligned}
2x - y &= -1 \\
x + 5y &= 3
\end{aligned}$$

4.

$$\begin{aligned} x - 2y &= 2 \\ 4x + 7y &= 6 \end{aligned}$$

5.

$$\begin{aligned} x + y &= 10 \\ x + 2y &= -4 \end{aligned}$$

Example. We now explore a method that will work with systems of equations in with many unknowns. First, we list some things that we can do to a system of linear equations.

1. The order of the equations can be switched.

   If we switch equations (1) and (2), we write

   $R_1 \leftrightharpoons R_2$.

2. We can multiply BOTH SIDES of any equation by a non-zero real number.

   If we multiply both sides of equation (3) by 1/4, for example, we write

   $1/4R_3 \mapsto R_3'$.

3. We can add a multiple of one equation to another equation.

   For example,

   if we add 3 times the first equation to the second equation and put the result as the new second equation, we write

   $3R_1 + R_2 \mapsto R_2$.

By keeping track of these operations, we are better prepared to spot-out our mistakes. Indeed, in solving linear systems of equations, humans tend to make arithmetic errors, and it is a good idea to keep track of the steps. The notation, R_1, R_2, R_3, and so forth, that we use to indicate the equations, stands for row number 1, row number 2, row number 3, and so forth. Soon we will encode all the operations in the equations as MATRICES— rectangular arrays of numbers.

The rows in the matrix will encode each equation. So the letter R stands for row.

For the sake of keeping track of the arithmetic, I will work the example above, but by using the THREE ELEMENTARY ROW OPERATIONS that are enumerated above.

Example. Once again consider the pair of lines

$$\begin{aligned} 2x - 3y &= 5 \\ 4x + 5y &= 6 \end{aligned}$$

Muliply both sides of eqn. 1 by -2 and add the result to eqn. 2:
$[2R_1 - R_2 \mapsto R_2']$

$$\begin{aligned} 2x - 3y &= 5 \\ 4x + 5y + (-4x + 6y) &= 6 + (-10) \end{aligned}$$

The resulting system upon simplification looks like:

$$\begin{aligned} 2x - 3y &= 5 \\ 11y &= -4 \end{aligned}$$

Multiply the second eqn. by (1/11):
$[\frac{1}{11}R_2 \mapsto R_2']$

$$\begin{aligned} 2x - 3y &= 5 \\ y &= -4/11 \end{aligned}$$

In the new system of equations multiply the second equation by 3 and add to the first: $[3R_2 + R_1 \mapsto R_1']$.
Upon simplification we get:

$$\begin{aligned} 2x &= 5 + (-12)/11 \\ y &= -4/11 \end{aligned}$$

Finally, we mulitply the first equation through by 1/2:
$[\frac{1}{2}R_1 \mapsto R_1]$

$$\begin{aligned} x &= \frac{1}{2} \cdot \frac{43}{11} \\ y &= -4/11 \end{aligned}$$

Thus the intersection point is
$(43/22, -4/11)$ which agrees with our previous calculation!

Our step-by-step goals in this process (for as many equations in as many unknowns as we like) are (1) eliminate the first variable from all subsequent equations, (2) eliminate the second variable from the 3rd, 4th, 5th etc, equations, and continue, then eliminate the last variable from all the equations above, the next to the last from those above it, and so forth. Occassionally, it is convenient to interchange the order of equations. This need for convenience is why we have row reduction rule (1).

Exercises.

In the original 5 systems of equations above, solve the system by performing elementary row operations to the equations.

Commentary.

While the step-by-step goals above seem pretty clear, the goals are not alway achievable. Not all pairs of lines in the plane intersect. Consider for example a pair of parallel lines. Or when considering more equations in more unknowns, one equation may be a consequence of the others, so that the intersection set among the equations is bigger than a point. Specifically, the intersection set between a pair of planes in 3-dimensional space usually is a line; look at where the wall meets the ceiling. The planes may not intersect;

consider the ceiling and the floor. Before we consider these anomolies, let us further condense the equations to matrices.

2.1.13 Converting systems of equations to matrices

In this section we revisit the same pair of lines and convert the lines to matrices. We illustrate the method of solution entirely on matrices. The reasons for converting our equations to matrices are as follows. The notation of matrices is more economical, and in this economical notation there a rich algebraic structure underlying the theory of solving systems of equations. This structure permeates our understanding of modern physics, and most applied areas of mathematics. Truely the importance of matrices in the mathematical models of the world cannot be underestimated.

Example. Once again consider the pair of lines

$$\begin{aligned} 2x - 3y &= 5 \\ 4x + 5y &= 6 \end{aligned}$$

We express this pair of equations as a rectangular array of numbers:

$$\left[\begin{array}{cc|c} 2 & -3 & 5 \\ 4 & 5 & 6 \end{array}\right]$$

Now we perform the same set of row operations in the same order to the matrices.

$$\left[\begin{array}{cc|c} 2 & -3 & 5 \\ 4 & 5 & 6 \end{array}\right] \overset{2R_1 - R_2 \mapsto R_2'}{\Longrightarrow} \left[\begin{array}{cc|c} 2 & -3 & 5 \\ (4-4) & (5+6) & 6+(-10) \end{array}\right] = \left[\begin{array}{cc|c} 2 & -3 & 5 \\ 0 & 11 & -4 \end{array}\right]$$

$$\overset{\frac{1}{11}R_2 \mapsto R_2'}{\Longrightarrow}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 5 \\ 0 & y & -4/11 \end{array}\right] \overset{3R_2 + R_1 \mapsto R_1'}{\Longrightarrow} \left[\begin{array}{cc|c} 2 & 0 & 5+(-12)/11 \\ 0 & 1 & -4/11 \end{array}\right] \overset{\frac{1}{2}R_1 \mapsto R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 0 & \frac{43}{22} \\ 0 & 1 & -4/11 \end{array}\right]$$

Thus the intersection point is $(43/22, -4/11)$ which agrees with our previous calculation!

Exercises

1. Apply the same sequence of operations to the original five systems two equations in two unknowns as a sequence of row operations to matrices.

2. Invent three systems of two equations in two unknowns and solve them using matrices.

2.1.14 Calculator Methods

The following set of instructions applies to the TI-83 calculator.

step 1 Turn on the calculator (9th row 1st column)

step 2 Hit the matrix key (3rd row 2nd column)

step 3 The NAMES item is highlighted use the arrow keys on the upper right to move the highlight to EDIT. Hit enter, and this should edit the matrix whose variable name is $[A]$.

step 4 We want to resize the matrix to be a (2×3)-matrix. [two rows and three columns]. type 2, enter, 3 enter.

step 5 Now edit the matrix. For example as above we have 2, enter, 3, enter, 5 , enter 4, enter, 5, enter, 6 enter. The 2,3 that appear of the bottom tells you that you just are editing the second row third column of the matrix. If you make an error, you can use the arrow keys to move the edit cursor into the correct entry.

step 6 Now we store the matrix by hitting 2nd quit. [Second, causes the yellow commands to function]. IT IS IMPORTANT TO QUIT THE MATRIX EDITOR.

step 7 Check your entry. Hit matrix button again, but this time leave the highlighted items on NAMES and A. Then hit enter. You should see an array of numbers as above, but without the vertical lines.

step 8 Hit matrix again, then move the cursor to MATH. Now use the down arrow until the item `rref` appears. This is item B on that menu screen. You can get there faster by scrolling up. Once there hit enter.

step 9 Now you have to tell the calculator the matrix, so go to matrix NAMES A and hit enter. Finally close the parentheses and hit enter again.

step 10 Now touch the math key (third row first column), and hit enter. This gives the "reduced row echelon form" with the entries as rational numbers.

Exercises

Work with your original five system of two equations in two unknowns and check your answers originally.

2.2 Linear equations in 3-unknowns

Example Consider the equation

$$2x + 3y + z = 6.$$

This equation represents a plane in 3-dimensional space. By the same analysis as we did with lines, we can see that the intercepts with the coordinate axes are the points $(3, 0, 0)$ —the x intercept, $(0, 2, 0)$ — the y-intercept, and $(0, 0, 6)$ — the z-intercept. In the illustration that I will draw for you, I will indicate how to depict the x, y, and z axes on a sheet of paper. The process always involves a right handed coordinate system. We illustrate the plane by drawing a triangle which is the intersection of the plane in question with each of the coordinate planes.

The coordinate planes have special equations. The (xy)-plane is the plane $z = 0$. The (yz)-plane has equation $x = 0$. The (xz)-plane has equation $y = 0$. The phrase that I use at this point is, "Everything is what it is, isn't it?" By which I mean, "The plane that has all possible values for x and y has its z value 0. This plane is perpendicular to the z-axis."

In our first set of exercises on planes, we will consider planes whose x, y, and z intercepts are all positive. I have tricks for rotating your coordinate pictures in the other 7 cases. In the given exercises try to pay attention to the scale along the axes so that relative lengths look right.

Thought Exercise. Why are there a total of eight cases among the signs of the intercepts?

Exercises.

Graph the planes:

1. $$x + y + 2z = 1$$
2. $$x + 2y + z = 6$$
3. $$2x + y + z = 6$$
4. $$x + 2y + 3z = 6$$
5. $$2x + 3y + z = 6$$
6. $$2x + 3y + z = 6$$
7. $$x + 3y + 2z = 6$$
8. $$3x + 2y + z = 6$$
9. $$3x + y + 2z = 6$$
10. $$2x + y + 2z = 1$$
11. $$2x + y + 3z = 1$$
12. $$2x + 3y + z = 1$$
13. $$x + 2y + 3z = 1$$
14. $$3x + 2y + 5z = 30$$

The next bit needs me to explain a little or to provide illustrations. After my explanation and illustrations, attempt the following:

Exercises

1. $$3x + 2y - 5z = 30$$

2. $$3x - 2y + 5z = 30$$

3. $$-3x + 2y + 5z = 30$$

4. $$3x - 2y - 5z = 30$$

5. $$-3x + 2y - 5z = 30$$

6. $$-3x - 2y + 5z = 30$$

7. $$-3x - 2y - 5z = 30$$

2.3 Parametric Form of a Plane

To parametrize a plane we need two free variables because a plane is 2-dimensional. That means to find a location in a plane we need an initial point on the plane, and two non-parallel directions that specify the "streets and avenues" in the plane. The point on the plane is like the center of town. The directions of the streets and avenues need not be perpendicular. I will show you how to get the parametric form in an example.

Example. Consider the plane

$$4x + 3y + 6z = 24.$$

We compute by inspection that the intercepts are $(6, 0, 0)$, $(0, 8, 0)$, and $(0, 0, 4)$. Now we solve for x in terms of y and z.

$$\begin{aligned} 4x + 3y + 6z &= 24 \\ 4x &= 24 - 3y - 6y \\ x &= 6 - \frac{3}{4}y - \frac{3}{2}z \end{aligned}$$

Next we take this last form and attach to it two redundant equations.

$$\begin{aligned} x &= 6 - \frac{3}{4}y - \frac{3}{2}z \\ y &= 0 + y + 0z \\ z &= 0 + 0y + 1z \end{aligned}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{-3}{4} \\ 1 \\ 0 \end{pmatrix} \cdot y + \begin{pmatrix} \frac{-3}{2} \\ 0 \\ 1 \end{pmatrix} \cdot z$$

From this last equation we see that $(6, 0, 0)$ is a point on the plane, and that the direction vectors are $(\frac{-3}{4}, 1, 0)$, $(\frac{-3}{2}, 0, 1)$. If we like we can rescale these direction vectors so that the entries are whole numbers. The vectors $(-3, 4, 0)$, $(-3, 0, 2)$ are parallel to the given vectors, and can be used just as well.

Exercises

For each of the planes in the previous two sets of exercises determine the parametic form of the plane by following the preceeding example.

2.3.1 The Vector Cross Product

The vector cross product in 3-dimensions assigns to a pair of vectors $\vec{v}$ and $\vec{w}$ a third vector $\vec{v} \times \vec{w}$ satisfies these properties:

- $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.
- The length of $\vec{v} \times \vec{w}$ is equal to the area of the parallelogram two of whose sides are the vectors $\vec{v}$ and $\vec{w}$. (A corner of the parallelogram is the point $(0, 0, 0)$). The other three corners are the end-points of $\vec{v}$, $\vec{w}$, and $\vec{v} + \vec{w}$).
- $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

We let $\hat{\imath} = (1, 0, 0)$, $\hat{\jmath} = (0, 1, 0)$, and $\hat{k} = (0, 0, 1)$ (The funny hats are used so that the symbols are reserved). By convention we say that $\hat{\imath} \times \hat{\jmath} = \hat{k}$. The convention is called a SIGN CONVENTION. It is a choice of a positive direction between two possible. The parallelogram determined by $\hat{\imath}$ and $\hat{\jmath}$ is just a square of area 1. The vector $\hat{k}$ is clearly perpendicular to $\hat{\imath}$ and $\hat{\jmath}$, and its length is 1. So far so good. To deal with the third property, we let $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.

The area property in general is really kind of tricky, but it turns out it follows from algebraic properties of the cross product. The specific algebraic rules are:

- The cross product distributes over $+$. Specifically,
$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}.$$
-
$$\begin{aligned} \hat{\imath} \times \hat{\jmath} = \hat{k}, &\quad \hat{\jmath} \times \hat{\imath} = -\hat{k}, \\ \hat{\jmath} \times \hat{k} = \hat{\imath}, &\quad \hat{k} \times \hat{\jmath} = -\hat{\imath}, \\ \hat{k} \times \hat{\imath} = \hat{\jmath}, &\quad \hat{\imath} \times \hat{k} = -\hat{\jmath}. \end{aligned}$$

Finally, $\hat{v} \times \hat{v} = 0$ for any vector v. This can be seen geometrically because the parallelogram is completely squished! There is an algebraic reason as well:
$$\vec{v} \times \vec{v} = -\vec{v} \times \vec{v}$$
(Here the factors in the second term were switched). The only thing that is equal to its negative is $\vec{0}$.

Example Ok, here I want to do a simple example first. But I want to work a lot of details. The plane $x + y + z = 1$ intersects the coordinate axes at $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. The parametric form of the plane is
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot y + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot z$$

The vectors $(-1,1,0)$ and $(-1,0,1)$ can be expressed as $(-1,1,0) = -\hat{\imath} + \hat{\jmath}$, and $(-1,0,1) = -\hat{\imath} + \hat{k}$. Now we compute

$$\begin{aligned}
(-1,1,0) \times (-1,0,1) &= (-\hat{\imath} + \hat{\jmath}) \times (-\hat{\imath} + \hat{k}) \\
&= (-\hat{\imath} + \hat{\jmath}) \times (-\hat{\imath}) + (-\hat{\imath} + \hat{\jmath}) \times \hat{k} \\
&= (-\hat{\imath}) \times (-\hat{\imath}) + \hat{\jmath} \times (-\hat{\imath}) + (-\hat{\imath}) \times \hat{k} + \hat{\jmath} \times \hat{k} \\
&= \vec{0} + \vec{k} + \vec{j} + \hat{i}
\end{aligned}$$

So the result of the cross product of the direction vectors on the plane is $(1,1,1)$. Observe this is the perpendicular vector to the plane. The length of this vector is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

One more observation is due here. The triangle formed by the intersection between coordinate planes and the plane $x+y+z = 1$ plane has area $\sqrt{3}/2$— it is $1/2$ of the area of the parallelogram.

Example This example comes from the plane $4x + 3y + 6z = 24$. Recall that the vectors $(\frac{-3}{4}, 1, 0)$ and $(\frac{-3}{2}, 0, 1)$ are parallel to the plane. We mimic the calculation above:

$$\begin{aligned}
\left(\frac{-3}{4}, 1, 0\right) \times \left(\frac{-3}{2}, 0, 1\right) &= \left(\frac{-3}{4}\hat{\imath} + \hat{\jmath}\right) \times \left(\frac{-3}{2}\hat{\imath} + \hat{k}\right) \\
&= \left(\frac{-3}{4}\hat{\imath} + \hat{\jmath}\right) \times \left(\frac{-3}{2}\hat{\imath}\right) + \left(\frac{-3}{4}\hat{\imath} + \hat{\jmath}\right) \times \hat{k} \\
&= \left(\frac{-3}{4}\hat{\imath}\right) \times \left(\frac{-3}{4}\hat{\imath}\right) + \hat{\jmath} \times \left(\frac{-3}{2}\hat{\imath}\right) \\
&+ \left(\frac{-3}{4}\hat{\imath}\right) \times \hat{k} + \hat{\jmath} \times \hat{k} \\
&= \vec{0} + \left(\frac{-3}{2}\right) \vec{k} + \left(\frac{-3}{4}\right) \vec{j} + \hat{i}
\end{aligned}$$

Exercises

For each of the planes above determine the cross-product between the direction vectors that you found in the parametric form. The resulting direction vector should be a multiple of the vector (A, B, C) if the plane has equation $Ax + By + Cy = R$.

Compute the length of the cross product vector. For example, if the vector is $(3, 2, 5)$, then its length is $\sqrt{3^2 + 2^2 + 5^2} = \sqrt{9 + 4 + 25} = \sqrt{38}$.

2.4 The Intersection Between Two Planes

The intersection between two planes will be a line in space. Since the line is in space and not in the plane, we will have to specify it in parametric form. In order to determine the line of intersection, we solve a system of 2 linear equations in 3 unknowns. First, I will show the equations. Next we will write it in matrix form. Then we will row reduce the matrix. In the first example, I do this by hand. In your homework you can use the calculator.

Example Consider the system of equations

$$\begin{aligned} x + y + z &= 1 \\ 3x - 2y + 2z &= 5 \end{aligned}$$

We write this in matrix form and perform the row reduction:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & -2 & 2 & 5 \end{pmatrix} \overset{-3R_1+R_2 \mapsto R_2}{\Longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -5 & -1 & 2 \end{pmatrix}$$

$$\overset{-1/5R_2 \mapsto R_2}{\Longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1/5 & 2/5 \end{pmatrix} \overset{-R_2+R_1 \mapsto R_1}{\Longrightarrow} \begin{pmatrix} 1 & 0 & 4/5 & 3/5 \\ 0 & 1 & 1/5 & 2/5 \end{pmatrix}$$

We obtain

$$\begin{aligned} x + \frac{4}{5}z &= \frac{3}{5} \\ y + \frac{1}{5}z &= \frac{2}{5} \end{aligned}$$

Rearrange the equations in much the same way as you did before:

$$\begin{aligned} x &= \frac{3}{5} - \frac{4}{5}z \\ y &= \frac{2}{5} - \frac{1}{5}z \\ z &= 0 + 1z \end{aligned}$$

Thus we obtain $\vec{\ell}(z) = (3/5, 2/5, 0) + z(-4/5, 1/5, 1)$. A point on the line is $(3/5, 2/5, 0)$ and the line goes in the direction $(-4/5, 1/5, 1)$.

Exercises

Determine the line of intersection between the two planes:

1.

$$\begin{aligned} x+y+z &= 1 \\ 3x-2y+2z &= 5 \end{aligned}$$

2.

$$\begin{aligned} x+2y+z &= -1 \\ x-2y+2z &= 3 \end{aligned}$$

3.

$$\begin{aligned} 2x+y+z &= 1 \\ 3x-y+2z &= 2 \end{aligned}$$

4.

$$\begin{aligned} x+y+2z &= 3 \\ x-2y-z &= -3 \end{aligned}$$

5.

$$\begin{aligned} 6x-3y+2z &= 7 \\ x-y-z &= 6 \end{aligned}$$

6.

$$\begin{aligned} 2x+3y+4z &= 4 \\ x+3y-3z &= 2 \end{aligned}$$

7.

$$\begin{aligned} 2x+y+2z &= 10 \\ x-y+z &= 15 \end{aligned}$$

8.

$$\begin{aligned} x+y+z &= 1 \\ 2x-y-z &= 3 \end{aligned}$$

9.

$$\begin{aligned} -4x+2y+z &= 6 \\ 6x-y+2z &= 8 \end{aligned}$$

10.

$$\begin{aligned} 5x-2y+3z &= 3 \\ x-3y-2z &= 4 \end{aligned}$$

Exercises

Determine the point of intersection among the three planes (use your caculator to compute the reduced row-echelon form:

1.

$$\begin{aligned} x+y+z &= 1 \\ 3x-2y+2z &= 5 \\ x+0y+z &= 3 \end{aligned}$$

2.

$$\begin{aligned} x+2y+z &= -1 \\ x-2y+2z &= 3 \\ x+y+3z &= 3 \end{aligned}$$

3.

$$\begin{aligned} 2x+y+z &= 1 \\ 3x-y+2z &= 2 \\ -2x+y+z &= 3 \end{aligned}$$

4.

$$\begin{aligned} x + y + 2z &= 3 \\ x - 2y - z &= -3 \\ x + y + z &= 3 \end{aligned}$$

5.

$$\begin{aligned} 6x - 3y + 2z &= 7 \\ x - y - z &= 6 \\ 2x + 2y + z &= 5 \end{aligned}$$

6.

$$\begin{aligned} 2x + 3y + 4z &= 4 \\ x + 3y - 3z &= 2 \\ x + 0y + z &= 5 \end{aligned}$$

7.

$$\begin{aligned} 2x + y + 2z &= 10 \\ x - y + z &= 15 \\ x + y + 0z &= 3 \end{aligned}$$

8.

$$\begin{aligned} x + y + z &= 1 \\ 2x - y - z &= 3 \\ 2x + y + 2z &= 2 \end{aligned}$$

9.

$$\begin{aligned} -4x + 2y + z &= 6 \\ 6x - y + 2z &= 8 \\ x - 2y + 2z &= 3 \end{aligned}$$

10.

$$\begin{aligned} 5x - 2y + 3z &= 3 \\ x - 3y - 2z &= 4 \\ 2x + y + 2z &= 3 \end{aligned}$$

We have finished our analysis of linear expressions. in two and three variable. There is much more to touch upon: For example symmetries and rotations of space, solving m equations in n unknowns, the general notion of vector spaces, sets of symmetries of space, etc. Certainly, a whole course could and should be devoted to linear phenomena.

Chapter 3

Quadratic Expressions

In this chapter, we will study quadratic expressions. These are expressions that involve terms with either x^2, y^2, z^2, xy, xz or yz, and combinations of these. Linear expressions provide straight line and other flat approximations to the real world. Quadratic expressions allow for certain types of roundness. Circles, spheres, ellipses, parabolas and hyperbolas all provide examples of quadratic expressions. When you throw a ball in the air, the shape of the path is a quadratic function.

3.1 The Basic Parabolas

We start from the basic quadratic function, $y = x^2$. This equation is a FUNCTION in that each value of x results in a unique value of y. (BTW, linear expressions in the form $y = Mx + B$, are also functions). Let us tabulate some values:

x	y
0	0
± 1	1
± 2	4
± 3	9
± 4	16

Exercise

On a good piece of graph paper, sketch a graph of the parabola $y = x^2$. Include more values, including fractional values for x. And know your set of squares: $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625\}$ These are the squares of the whole numbers from 1 to 25. Observe that the difference between successive squares is alway an odd number, and that the odd number differences go $1, 3, 5, 7, \ldots$. So if you know a pair of successive squares like $15^2 = 225$ and $16^2 = 256$ you can compute the difference $256 - 225 = 31$, and get $17^2 = 256 + 33 = 289$.

BTW, I used this trick as I was typing up the list, and checked with the calculator. Knowing by heart this set of squares can be very helpful in other numeric calculations.

3.1.1 example

The expression $y = (x - 1)^2$ is also a parabola. We can easily graph this by shifting the values in our table above. We want the value within the parentheses to be $0, \pm 1, \pm 2$, etc. So for example, $x - 1 = 0$ when $x = 1$.

1. Horizontal Motion
2. VerticalMotion
3. Vertical Stretching
4. Horizontal Stretching
5. Combining Motion

3.1.2 Completing the Square

1. Every rectangle is the difference of two squares
2. Quadratics with no dilation
3. Including the Dilation

4. Finding the Vertex

5. Finding the y-intercept

6. Finding the x-intercept

3.1.3 Going off on a Tangent

1. Average rate of change

2. linear Interpolation

3. Instantaneous Rate of Change

4. Computing the Equation of the Tangent

5. The Slope of the Tangent at the Vertex

3.1.4 The Area Under the Parabola

3.2 Quadratic curves

3.2.1 Circle

3.2.2 Ellipse

3.2.3 Hyperbola

3.3 Quadratic Surfaces

3.3.1 Sphere

3.3.2 Parabolic Cone

3.3.3 Hyperboloids

3.3.4 Critical Phenomena

3.3.5 Other Surfaces

3.3.6 Evolution

3.4 Absolute Values

3.4.1 Positive/Negative, Up/Down, Right/Left

3.4.2 Analogous Formulas

3.4.3 Solving Inequalities

3.4.4 General Graphing

3.4.5 Where is the Tangent

3.5 Polynomial Functions

3.5.1 Factored Form

3.5.2 Cubic Functions and Critical Phenomena

1. Factoring Using the Rational Root Theorem

2. Approximating Real Roots

3. Complex Roots

3.5.3 Multiplying

3.5.4 Synthetic Division

3.5.5 Tricks of the Trade

3.5.6 Critical Phenomena

3.5.7 Area Formulas

3.6 Rational Functions

3.6.1 Key Examples

3.6.2 Solving Inequalities—sign charts

3.6.3 Long-term Behavior

3.6.4 Vertical Asymptotes

3.6.5 More General Curves

3.6.6 Critical Phenomena

3.6.7 Infinite Length but Finite Area

3.7 Exponential Functions

3.7.1 Money

3.7.2 The Rules of Exponents

3.7.3 Arithmetic without Calculators

3.7.4 The Main Applications

3.7.5 The Rate of Change

3.7.6 The Area under a Curve

3.8 Trigonometric Functions

3.8.1 Fundamental Angles

3.8.2 SOHCAHTOA

3.8.3 Trigonometric Graphs