Mathematical Content in Elementary and Secondary Education

J. Scott Carter

December 28, 2007

This article is an edited version of a longer essay with the same thesis: *The mathematical training students receive in elementary, middle, and high school suffers because the education community is not engaged in a dialogue with the greater community of mathematicians.*

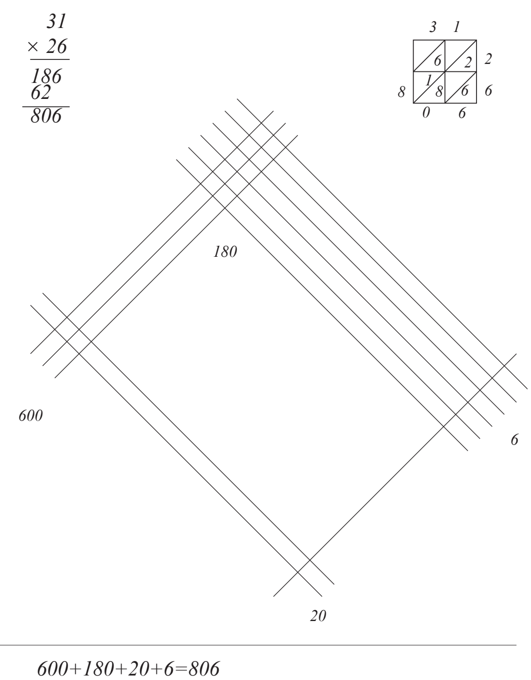
The core debate centers upon: (1) "algorithm versus understanding," (2) "an inquiry based approach versus rote learning," (3) "pure versus applied," and (4) "number problems versus word problems."

School teachers neither ask mathematicians how we learned how to do mathematics, nor how we think about mathematics. Either they already know the answer, or more dangerously, they think that they do. Twain said, "It aint what you dont know that gets you into trouble. Its what you know for sure that just ain't so." Research mathematics involves re-examining assumptions, performing rote calculations and thinking deeply about the nature of number, logic, and relationships among quantities. When mathematicians work, we learn to understand basic material. This understanding translates into more concise descriptions, more colorful metaphors, and more evocative examples.

Algorithm versus understanding

The first step in learning multiplication is to memorize the multiplication tables from 1 to 10. If from a young age a child is taught the idea of counting by twos, counting by threes, up through counting by eights, nines, and tens, then that child has an understanding that three groups of four is twelve. She can count, "four, eight, twelve," and she might even visualize a four by twelve array. Does a child routinely and unconsciously convert a multiplication problem into area or volume to try and see the product? When walking to St. Ives, does she arrange the wives, sacks, and cats into a cubical array with 343 elements? Perhaps, perhaps not; only one was walking to St. Ives.

The essence of the multiplication of one digit numbers is to have memorized the table. To multiply a pair of two digit numbers, memorization of the tables is combined with one of three standard methods for adjusting for place value. If the digits are lined up in 26 times 31 (a favorite example on youtube videos), then the algorithm yields 186+62*= 806. Alternatively, six lines plus two line perpendicularly cross three lines plus one line, in a diagonal fashion, and the crossing points are tallied with their position in the plane serving as a symbolic place holder, or a lattice is drawn to hold the places. No matter what, the algorithms involve a clever application of the *distributive law*: $(a+b)c = ac+bc$, or precisely $(10a + b)(10c + d) = 100ac + 10(bc + ad) + bd$. The three methods are illustrated in the next figure which necessarily lacks information since it illustrates the end product of three processes rather than the processes themselves.



To understand arithmetic is *to be able to apply the axioms of algebra to solve an arithmetic problem.* By understanding, does one mean that the student understands that she is using the distributive law? Did any of us understood the applications of routine algorithms for multiplication as an application of the distributive law when we first learned to multiply? Does a third grader internalize multiplication as an expression of area?

If teachers do not developing drill problems in the use of *some* algorithm, they are missing the opportunity to teach deeper understanding. Adults in industrialized nations routinely use calculators to perform mundane arithmetic, but the methods that are employed in performing arithmetical calculations are the same processes employed in advanced calculation in the scientific and mathematical world. Professionals learn how to compute with a multilinear forms by having practiced arithmetic. An argument is given that "most" people do not need that level of understanding. No! Most white collar workers need to understand logical functionality in order to maintain their electronic gadgets. Many people will be working within the computer industry, writing code or designing circuits, for example. Search engines use algorithms that can be understood in terms of linear functions: functions that distribute over addition as multiplication does. Anyone who is involved in problem solving will be engaged in mathematical activity.

School teachers must understand mathematics at a deeper level than their students do if the latter are to understand at all. Algebra at the level of junior high school and high school is the precise axiomatization of arithmetic that was learned in grade school. *Mathematical understanding is to know the collection of axioms in which computations take place, to be familiar with a large number of consequences of those axioms, and to be able to derive and verify further consequences in order to facilitate computations.*

The beginning high school algebra student should understand that the general multiplication of binomials yields generic arithmetic facts. For example, several talented mentalists use the difference of squares theorem $(a+b)(a-b) = a^2 - b^2$ to compute products of proximate two digit numbers in their heads. They have memorized the squares from 1 to 100, and have developed the ability to subtract a pair of squares rapidly and mentally. In fact, one does not "memorize" but uses algebraic and mnemonic tricks to derive the needed facts.

The non-mathematical reader who skimmed over the preceding paragraph and thought, "That is beyond me. I can use a calculator," Is expressing a dangerous attitude. When one says "I am sorry, I was never very good at math," is the proclamation a matter of pride or shame?

A child can become engaged in an inquiry-based activity based on the computation of squares and products as differences of squares. In this way arithmetic leads into algebra. Therefore, a rigorous application of algebra to arithmetic could help young people internalize algebraic facts. But in order for the child to make the connections, her teacher must already know the connection.

I, and most of my colleagues, look for common factors before embarking upon a problem of long division. Dividing by a large number, or more precisely one with many digits, can often be avoided. Long division, inspecting for common factors, recognizing divisibility, all are items that translate into algebraic skills. Divisibility tests for three, seven, nine, and eleven are easy to implement and they develop mental arithmetic skills.

In addition and subtraction there are two main difficulties: borrowing and carrying. To learn these skills try subtracting squares, eg, 25-9, 81-49, etc, and use these results to compute products as outlined above (8×2, and 16×2, respectively). Finally, before adding long columns of numbers, try adding from left to right, and thereby focus on the digits of most import.

What is meant by "understanding" carry digits? Certainly, place value is foremost in the mind. But there are deeper understandings related to modular (clock) arithmetic and odometer functions. These can be formulated in very abstract terms and are the basis for intricate algebraic constructions called cohomology theories.

In the sixties, it was thought that set theory should be incorporated in the curriculum. This addition of notation and conception helped me and several of my colleagues become mathematicians. And even though Tom Lehrer's song, "The New Math," still tickles my funny bone, there is no real harm in teaching abstraction to young people, *as long as the teachers understand the abstraction.*

So children could and should learn some aspects of modular arithmetic. Young adults should hear that the logarithm formalizes the analogy between multiplication and addition. The ideas of a group (presented, for example, as the set of transformations in geometry that preserves congruence), of a vector space (as a method performing arithmetical operations in ordinary space), of a polynomial ring (with its own division algorithm and the ancillary notion of field extension), are accessible to high-school students.[1] And introducing these abstractions aides understanding. But the same caveat as above applies.

Rote learning sometimes is superior to understanding. Consider for example the sine of sixty degrees. If you thought for a moment, and realized the answer is $\sqrt{3}/2$, then you learned this by rote. If you needed to set your calculator to determine an approximate numeric value of 0.866, then you do not remember trigonometry to extent that a college teacher would call understanding.

Understanding why something is true involves proof. Knowing something is true involves rote learning. Both are valuable. Many proofs are done so frequently that they, themselves are matters of rote learning. A musician learns by repeatedly playing the same piece until her ears are adept. The weight-lifter builds muscle mass by repetition; the mathematics student builds strong brain power by having worked exercises repeatedly. Learning mathematics, like learning anything requires repetition.

Do school students understand fractions? The property,$(a/b = c/d)$ if and only if $(ad = bc)$, is the definition of equality of two fractions. A fraction, as a formal ratio, is an equivalence class of sets of pairs of integers; the arithmetic laws for fractions are axiomatic.

Yes certainly, fractions less than one can be slices of pizza, and the addition and subtraction of fractions can be facilitated by further subdividing the pizza cuts. Especially here by understanding, we are not asking for an axiomatic understanding, but a geometrical understanding. Nevertheless, the division of fractions is best understood in a rote fashion. Food descriptions for fractional division leave me cold. Even though I have reconstructed a food-theoretic description of dividing one fraction by another, the metaphor just does not stick to my ribs as nicely as the rote algorithm: *to divide fractions, multiply by the reciprocal of the divisor, cancel like factors between numerator and denominator, and multiply the resulting factors in both numerator and denominator.*

Rote computation and routine implementation of algorithms facilitates understanding. A qualified teacher of mathematics knows enough mathematics in order to recognize whether or not the implementation of an alternative algorithm will result in correct results. Such a qualified teacher will be able to prove or disprove theorems as needed. A fully qualified teacher understands mathematics via the metaphors with which it is presented and via the axiomatic method. Both a sufficient supply of metaphors and an internalization of the axioms on the part of the teacher are necessary for the teacher to be effective.

Inquiry based learning

The first time that I seriously engaged in anything that I would classify as "inquiry based" learning in mathematics was as I began the research on my PhD. I was given a paper, told to read and understanding it, and see if I could generalize it.

No that is not fair. When I became curious about the shape of her diamond ring, my mother showed me how to use a protractor to draw the complete graphs on 8 and 9 vertices using regular polygons and connecting every pair of vertices. I learned that the number of lines decreased in a triangular fashion, and the sequence of triangular numbers became hard-wired in me. There was no mathematician around to show me that I was drawing a

[1] Complete and detailed definitions of these terms can be found at Wikipedia

fundamental object in higher dimensional space: the n-dimensional simplex. So perhaps a well-guided inquiry based learning experience could have lead me to discover Pascal's triangle, and the combinatorics of inclusion/exclusion. Who knows, if I had combined these ideas with an understanding of carry digits, I might have re-discovered homology theory at age 13.

Inquiry-based learning in mathematics is hooey, *unless a teacher is well-trained to see when the child has really found something, and unless the teacher knows where to find the resources to facilitate that inquiry.* How will a non-mathematically inclined teacher explain to first grader that $2 - 5 = -3$? Or will she exclude negative numbers from the cultural realm of the child, all the while wondering if she has enough money to pay the mortgage this month?

Pure versus applied mathematics

I once was in a meeting with several members of the local school board. We were developing an academy for the benefit of a functionally segregated school to help attract bright students. I spoke first, "To attract students to the mathematical sciences, we need to promote the truth and beauty of the subject." My scientifically trained colleague retorted later in the meeting, "Mathematics is better learned through application." Some of the women in the room had wide hips. Some were outright obese. My standard reply to the beauty versus utility debate is crude but effective: If utility is the only measure, then most men would be attracted to woman with wide hips. I need a metaphor that is less offensive to calorie challenged women: *If utility is the only measure, why are peahens attracted to peacocks with colorful tail feathers?*

The focus of this school's science academy is earth science. Happily, students will learn about water quality by digging a well. They will learn about biological systems by fish farming. Verily, the world needs more environmentalist, but also the number of African-American mathematical scientists who possess Ph.D.s is embarrassingly low.

The excellence with which African-Americans pursue the arts indicates that they will accept arguments to pursue mathematics for the sake of beauty. Every mathematician whom I know was attracted to the subject because of its intrinsic beauty. Utility is an aspect of that beauty. My high-school geometry teacher, who incidentally was African-American, always asked us, "Isn't this beautiful stuff, class?" Euclidean geometry is universally acknowledged as a paradigm for mathematical beauty. And equally importantly, it is what we use when we are building houses, moving furniture, measuring the height of buildings and trees indirectly, and solving virtually any problem in mechanical engineering.

Euclid's "Elements" is also the prototypical applied mathematics text. Euclid tells us: under these assumptions, we have those consequences. Muhammad ibn Musa al-Khwarizmi's ninth century text, "The Compendious Book on Calculation by Completion and Balancing" is also an applied mathematics book that, according to Wikipedia, covers techniques for solving linear, and quadratic equations, and "deals with complicated Islamic rules of inheritance." Even the most esoteric mathematical treatise that is being written today will likely find application in the next few generations.

Mathematics permeates modern society. Social and financial interactions are intermediated by communication devices such as the telephone and the world wide web, whose functionality depends on mathematical algorithms. Signals are transmitted and received on parabolic dishes; credit card numbers are encrypted using public-keys. Traffic flow on public thoroughfares in modern cities can be similarly routed via “timed” traffic lights. A mathematical metaphor demonstrates that traffic on busy highways is akin to digital traffic on the internet. Only through mathematics, can the logic gates that route traffic be analyzed. People who choose not to learn mathematics resign themselves to a life of tyranny. Information is power. Mathematical information is used to compute mortgages, insurance premiums, and costs of building materials.

Nonetheless, mathematical scientists are altruistic to a fault; we want people to understand our field and to learn our techniques. Time and time again, mathematics has been shown to give incredibly accurate models of the universe —so accurate that human kind can fly to and from the moon, launch satellites, and control electrons to such an extent that they can do arithmetic for you.

The mathematical metaphors that are taught in school should clearly include both pure and applied type problems. But a data set with more than two points cannot illustrate a straight line, and a data set that illustrates a linear relation illustrates nothing if the primitive geometric concept of a line is not understood beforehand. We must transcend the pure-applied dichotomy in mathematics.

Number problems and story problems

The concept of a number is that of an adjective. We cannot truly speak of three, but we can speak of three apples, three inches, three acres, three liters of water, or a three ohm resistor. Similarly, we can not speak of yellow, blue, and red. Yet both numbers and colors have rules of combination; blue and yellow make green; three cubed is twenty-seven. We deal with the adjectives as if they are things, we see patterns in the manipulation, and we axiomatize the patterns as relationships among variables. To show that relationships hold, we teach with specific models and work to abstract “two apples plus three apples is five apples” into $2+3=5$. Children accept abstraction since the whole world is an abstraction to them. The arithmetic problems that a child is to work are designed to (1) decouple the number from the objects involved, (2) provide an adequate set of calculations so that the arithmetic is internalized, and (3) provide patterns that can also be internalized. Subsequently, when the pattern is articulated and explicated, the child recognizes it as a truth.

There is another step. The truth that has been discovered via the pattern must be demonstrated to be true in all possible cases. This is the role of proof. In mathematics, we make educated guesses. We recognize patterns, and subsequently, we use logic to show that the patterns are permanent. Proofs keep us honest, and they are everlasting. So even though there are non-Euclidean geometries and a variety of algebraic systems, Euclid and Al-Khwarizmi remain fundamental sources for *the particular axiomatic system* that fits the world of ordinary perception.

Rote exercises that involve only numbers, only symbols, or a mixture of numbers and symbols provide the strength to compete in the bigger game. Story problems return the mathematics away from the adjectives and back towards the nouns. Good story problems are like good works of fiction. Human nature is explained clearly by Jane Austin, Joseph

Conrad, Toni Morrison, and Mark Twain. A clear ink sketch with broad strokes, simple lines, and empty space can characterize without caricature. Similarly, word problems and story problems in which numbers are chosen to work out nicely help teach the essential application of the mathematical concept.

A recent trend in problem writing is to use real numbers and real data to help teach the relevance of the mathematics. The problem with that idea is that real numbers are never nice. Arithmetical computations get in the way of deep understanding. This is an arena in which the calculator also gets in the way. When real numbers are used, and routine computations are entered into the calculator, then it is out-and-out impossible for the student to determine if the answer is correct since the intermediate steps contain nothing familiar.

It is true that if you want to determine the height of a real tree by an indirect measurement, you could walk 14.5 meters away from the base of the tree, use a surveyors' theodolite that is situated on a tripod at a height of 1.68 meters, and measure an angle of 56 degrees in the line of sight to the top of the tree. Then the tree would be 23.2 meters tall. This tree (albeit imaginary) is not nearly as interesting as the imaginary tree whose angle of inclination is 53 degrees when measured by a theodolite on a 2 meter tall tripod that is situated 15 meters from the base of the tree. The latter tree is nearly 22 meters tall since the tangent of 53 degrees is roughly 4/3. The latter tree is more interesting because the angle is chosen to simulate a $3, 4, 5$ triangle, and its distance is exactly 15 meters from the base of the tree; the 3 in the denominator cancels with the 3 in the 15. The numbers in the latter problem work out, and the student working the problem gets the idea that she got the correct answer *because the numbers work out nicely.*

The second problem was easier to write, it is easier to grade, and it characterizes the computation. Had the angle of inclination been $60°$, the problem would have been more meaningful still. In real life, one cannot simply walk 15 meters away from the tree, there may be an ants' nest or a mud puddle where one wants to locate the tripod. But in story problems, there is fiction built into the problem because the stories are parables that teach mathematical lessons. The stories inter-relate and characters from one story or another weave in and out of the other stories.

That is why fractions are thought of as pieces of pizza. The absurdity of the story when strange fractions occur makes the story problem delightfully silly. "Esmerela cut her tofu and onion pizza into twelfths and has five pieces left. Zenobia cut her mushroom and radish pizza into eighths and has three pieces left. What fraction of a pizza do they have between them? ANS: 19/24. Do they have more or less than three quarters of a pizza between them? ANS: yes.

So fractions are metaphorically pieces of pizza. The distributive law says that 3 apples plus 5 apples is 8 apples, or $3a + 5a = (3 + 5)a = 8a$. Multiplication refers to area, and any rectangle can be expressed as the difference of two squares. Tangents of angles are used to measure trees. Right triangles of type $3, 4, 5$ are easy for computations; $30 - 60 - 90$ triangles are interesting as are isosceles right triangles. These stories are part of our culture.

The final two personal stories demonstrate clearly that there is a problem with the mathematics taught is schools. My colleagues and I have a zillion[2] more.

A beginning college freshmen asked in class if she could work with decimals because

[2] a large but indeterminate number

fractions confused her. I told her I didn't understand: Why was the decimal representation of 1/7 more meaningful to her than the fraction was? Did she enjoy working with infinite repeating decimals, $0.142857142857\ldots$? She didn't get the joke, and she didn't stay in my class. It is a shame; there was a lot that she needed to be taught. I guess she thought I was being sarcastic. I was being sincere.

A man was staring up at my office building at 5 PM. It turns dark at about 5 during winter since I live on the eastern end of the time zone. His presence there at that time seemed strange. I interrupted his train of thought; I apologized. He was counting the bricks from the bottom of the building to the top. He told me that there are nine bricks per two feet, and he was estimating the height of the building since he was going to bid on a contract to wash and seal the exterior. I didn't ask him why he didn't use a theodolite. The building is three and a half stories, so counting bricks does not seem easy. I wondered if the building's height was not a matter of public record; we are a state school. But I learned a nice mathematical fact from the brick counter — one that will appear in a problem involving a building, a theodolite, the tangent of an angle, and the distance to the observer. I hope that I can write the problem to make the numbers turn out.

Mathematicians need to be invited to the table when issues of content are brought up in public school policy. More precisely, issues of content must be the framework of future discussions. Elementary school teachers and high school teachers are well-trained in classroom management, lesson planning, and reporting functions. But their content training is lacking if they should say, "school kids don't need to know that." Children should be encouraged to learn what adults already know. Children need to learn mathematics at or above their capacity. The mathematical training of young people must focus on content and should be made in consultation with professional mathematicians as well as professional scientists and engineers.