Ping pong balayage and convexity of equilibrium measures

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Abstract

In this paper we establish that the density of the equilibrium measure of finitely many intervals for both logarithmic and Riesz potentials is convex. The main tool is a ping-pong balayage technique. Similar result is obtained for finitely many arcs on the unit circle. Applications to external field problems and constrained energy problems are presented.

1 Introduction and Main Results

In recent years the theory of logarithmic potentials with external fields has played a vital role in solving some long standing problems in orthogonal polynomials, random matrices, integrable systems, etc. The intertwining relationship between these various areas and logarithmic potentials is well established in several monographs, for example [1], [10], [17], [21], [22], [25], [24], as well as many research articles cited therein. And yet, even in the classical problem of no external field some major geometric properties remain elusive.

For example, in a companion paper [5] we establish that harmonic measures (and in particular equilibrium measures) of compact sets on the real line and the unit circle have essentially log-convex density. In this work we extend this property for M. Riesz $s$-equilibrium measures (see Theorem 1.1 below). Motivated by our applications, we focus on the case of compact sets that are unions of finitely many intervals on the real line or finitely many arcs on the unit circle (the general case may be derived as in [5] by an approximation process).

The motivation for this work comes from applications to external field energy problems and constrained energy problems. The determination of the nature of the equilibrium support is a critical part in finding the solution in these problems. Theorems 1.9, 1.10, and 1.12 address these applications.

1.1 Convexity of logarithmic and Riesz equilibrium measures

First, let us introduce the potential-theoretical preliminaries necessary to state our results. In what follows a measure refers to a positive measure. Let

$$k_s(x,y) := \begin{cases} 
\frac{1}{|x-y|^s} & 0 < s < 1 \\
\log \frac{1}{|x-y|} & s = 0
\end{cases}$$

(1.1)

Given a finite Borel measure $\mu$ in the complex plane $\mathbb{C}$ and a number $s$, $0 < s < 1$, we denote the Riesz $s$-potential and the Riesz $s$-energy of $\mu$ with

$$U^s_\mu(z) := \int k_s(x,y) \, d\mu(x), \quad I_s(\mu) := \int \int k_s(x,y) \, d\mu(x) \, d\mu(z).$$

(1.2)

The $s$-energy of a compact set $K \subset \mathbb{C}$ and its $s$-capacity are defined as

$$W_s(K) := \inf \{ I_s(\mu) : \text{supp}(\mu) \subset K, \, \mu(K) = 1 \}, \quad \text{cap}_s(K) := 1/W_s.$$ 

(1.3)
A probability measure \( \mu_{s,K} \) supported on \( K \) such that \( I_s(\mu_{s,K}) = W_s(K) \) is called an \( s \)-equilibrium measure of \( K \). If \( \text{cap}_s(K) > 0 \), there exists unique \( s \)-equilibrium measure \( \mu_{s,K} \) characterized completely by the equation

\[
U_s^{\mu_{s,K}}(z) = W_s(K) \quad \text{q.e. on } K,
\]

where q.e. (quasi-everywhere) means that the exceptional set is of \( s \)-capacity zero (see [16, §2]). From the maximum principle and (1.4) one derives that

\[
U_s^{\mu_{s,K}}(z) \leq W_s(K) \quad \text{on } \mathbb{C}.
\]

We should note that the restriction \( 0 < s < 1 \) is prompted by the fact that \( K \) will be chosen as a subset of the real line or the unit circle.

In the classical case \( s = 0 \) we omit the subscript \( s \) and refer to the similarly defined quantities as logarithmic potential, logarithmic energy, Robin constant, logarithmic capacity \( (\text{cap}(K) := e^{-W(K)}) \), and (logarithmic) equilibrium measure respectively. Similar characterization as equation (1.4) holds true.

In what follows let \( E \) be a compact set on the real line \( \mathbb{R} \), respectively the unit circle \( \mathbb{T} \), which is union of finitely many intervals, respectively arcs, i.e.

\[
E = \bigcup_{j=1}^n [a_j, b_j]
\]

where \( a_1 < b_1 < \cdots < a_n < b_n \). In the case of unit circle we may assume without loss of generality that \( 0 = a_1 < b_n < 2\pi \), and in this case we let \( [a_j, b_j] := \{ z = e^{i\theta} : a_j \leq \theta \leq b_j \} \).

Our first result deals with the \( s \)-equilibrium measure of the set \( E \).

**Theorem 1.1** Let \( E \) be the union of finitely many compact intervals on the real line or closed arcs on the unit circle (see (1.6)). Then for any \( 0 \leq s < 1 \), the \( s \)-equilibrium measure \( \mu_{s,E} \) is absolutely continuous with respect to the Lebesgue measure and its density is strictly log-convex on every subinterval/subarc of \( E \).

**Remark 1.2** We recall that a positive function \( f \) is log-convex on an interval \( (a,b) \) if \( \log f(x) \) is convex on that interval. Log convex functions are also convex, so log-convexity of the \( s \)-equilibrium density implies mere convexity. We note that sum and multiplication of log-convex functions are log-convex. Also log-convexity is preserved by parametric integration.

**Remark 1.3** To illustrate the difficulties when trying to prove Theorem 1.1 directly, let us consider the case when \( E \subset \mathbb{R} \). Then the density with respect to the Lebesgue measure of the (logarithmic) equilibrium measure of \( E \) is given by (see [24, Lemma 4.4.1], [26, Eqs. (2.4) and (2.8)])

\[
\frac{d\mu_E(t)}{dt} = \frac{\prod_{j=1}^{n-1} |t - \tau_j|}{\pi \prod_{j=1}^{n} (|t - a_j||t - b_j|)^{1/2}},
\]

where the numbers \( \tau_j \) lie in the gaps \( (b_j, a_{j+1}) \), \( j = 1, 2, \ldots, n - 1 \), and are uniquely determined by the system

\[
\int_{b_i}^{a_{i+1}} \frac{\prod_{j=1}^{n-1} (t - \tau_j)}{\prod_{j=1}^{n} (|t - a_j||t - b_j|)^{1/2}} \, dt = 0, \quad i = 1, 2, \ldots, n - 1.
\]

In order to show that the density in (1.7) is log-convex we would have to prove that if \( \{\tau_j\} \) satisfy (1.8), then the following inequality holds

\[
\sum_{j=1}^{n-1} \frac{1}{(t - \tau_j)^2} \leq \frac{1}{2} \sum_{j=1}^{n} \left( \frac{1}{(t - a_j)^2} + \frac{1}{(t - b_j)^2} \right), \quad \text{for all } t \in E.
\]

Clearly, if \( \tau_j \) is too close to \( a_j \) or \( b_{j+1} \), this inequality may be violated. In [3] the inequality was proven by direct approach for the case of two intervals/arcs (the case of one interval/arc is trivial). Even in this case the technical difficulties are significant. We don’t know of a formula similar to (1.7), (1.8) when \( 0 < s < 1 \) and \( n \geq 2 \) (for \( n = 1 \) see (2.1) below).
Remark 1.4 We also note that the result on the unit circle is not a simple transformation of the result on the real line.

1.2 External field problem and constrained energy problem

Next, we introduce the external field problem (also known as weighted energy problem) and the constrained energy problem. Let $F$ be a closed set in the complex plane and $Q(x)$ be a lower semi-continuous function defined on $F$. It is called admissible external field if the set on which $Q$ is finite is of positive $s$-capacity and if $Q$ satisfies some suitable growth condition, say $Q(x) - \log |x| \to \infty$ as $|x| \to \infty$ (in case $F$ is unbounded). The weighted energy of a measure $\mu$ supported on $F$ is given by

$$I_{s,Q}(\mu) = I_s(\mu) + 2 \int Q(x) \, d\mu(x). \quad (1.10)$$

Under the admissibility assumptions there is unique minimizing probability measure $\mu_{s,Q}$ such that

$$I_{s,Q}(\mu_{s,Q}) = V_{s,Q} := \inf \{ I_{s,Q}(\mu) : \text{supp}(\mu) \subset F, \mu(F) = 1 \}. \quad (1.11)$$

Finding this minimizer is known as the external field problem. The measure $\mu_{s,Q}$, called $s$-extremal measure (or equilibrium measure) associated with $Q$, satisfies the variational inequalities

$$U_s^{\mu_{s,Q}}(z) + Q(z) \geq F_{s,Q} \quad \text{q.e. on } F, \quad (1.12)$$

and

$$U_s^{\mu_{s,Q}}(z) + Q(z) \leq F_{s,Q} \quad \text{on supp}(\mu_{s,Q}), \quad (1.13)$$

where $F_{s,Q} = V_{s,Q} - \int Q(x) \, d\mu_{s,Q}(x)$. For later reference we note that (1.12) and (1.13) imply the equation

$$U_s^{\mu_{s,Q}}(z) + Q(z) = F_{s,Q} \quad \text{q.e. on supp}(\mu_{s,Q}). \quad (1.14)$$

In the case when $F$ is regular w.r.t. Dirichlet problem (which is true if $F$ satisfies (1.6)) and $Q$ is continuous, the inequality (1.12), and hence equality (1.14) hold everywhere.

The variational inequalities completely characterize the $s$-extremal measure among measures with finite energy, i.e. if there is another probability measure with finite energy $\lambda$ supported on $F$ that satisfies (1.12) and (1.13) with some constant $C_\lambda$ instead of $F_{s,Q}$, then $\lambda = \mu_{s,Q}$ (and of course $C_\lambda = F_{s,Q}$). For the logarithmic case see [22, Theorems I.1.3 I.3.3] and for the Riesz case see [27, Theorem 1 and Proposition 1].

The constrained energy problem takes the generalization one step further (see [20], [12], [28]). Suppose that $\sigma$ is a fixed measure with $\sigma(F) > 1$. Without loss of generality we may assume that $\text{supp}(\sigma) = F$. The constrained energy problem is to determine a minimizer of (1.11), subject to the additional constraint that $\mu \leq \sigma$, meaning that $\sigma - \mu$ is a positive measure. Such a minimizing probability measure exists and is unique. We call it constrained $s$-extremal measure associated with external field $Q$ and constraint $\sigma$ and denote it by $\lambda_{s,Q}^\sigma$, i.e. we have

$$I_{s,Q}(\lambda_{s,Q}^\sigma) = \inf \{ I_{s,Q}(\mu) : \mu \leq \sigma, \mu(F) = 1 \}. \quad (1.15)$$

The variational inequalities that characterize uniquely $\lambda_{s,Q}^\sigma$ take the form

$$U_s^{\lambda_{s,Q}^\sigma}(z) + Q(z) \geq F_{s,Q}^\sigma \quad \text{q.e. on supp}(\sigma - \lambda_{s,Q}^\sigma), \quad (1.16)$$

and

$$U_s^{\lambda_{s,Q}^\sigma}(z) + Q(z) \leq F_{s,Q}^\sigma \quad \text{on supp}(\lambda_{s,Q}^\sigma). \quad (1.17)$$

Of particular interest in our case will be the case when $Q(x) \equiv 0$ and $U_s^\sigma(x)$ is continuous. Then, as shown in [11] for logarithmic potentials, the constrained energy problem reduces to a (usual, not constrained) external field problem. Indeed, even for Riesz potentials, using the maximum principle we extend inequality (1.17)
to the entire complex plane. The continuity of $U^\sigma_{\lambda}(x)$ implies the continuity of $U^{\lambda\sigma}_{\lambda}(x)$ as can be seen from the representation

$$U^{\lambda\sigma}_{\lambda}(x) = U^\sigma_{\lambda}(x) - U^{\sigma-\lambda\sigma}_{\lambda}(x),$$

and the lower semicontinuity of the potentials of measures. This shows that (1.16) holds everywhere on $\text{supp}(\sigma - \lambda\sigma_{\lambda,Q})$. The variational inequalities (1.16) and (1.17) can be written as

$$U^\sigma_{\lambda}(z) - U^{\sigma-\lambda\sigma}_{\lambda}(z) \leq -F^\sigma_{\lambda}$$

on $\text{supp}(\sigma - \lambda\sigma_{\lambda,Q})$, (1.18)

$$U^\sigma_{\lambda}(z) - U^{\sigma-\lambda\sigma}_{\lambda}(z) \geq -F^\sigma_{\lambda}$$

on $\mathbb{C}$.

Therefore, by the uniqueness of the external field problem

$$\frac{\sigma - \lambda\sigma}{\|\sigma\| - 1} = \mu_{s,Q}, \quad \text{where} \quad Q(z) = -\frac{1}{\|\sigma\| - 1}U^\sigma(z).$$

(1.20)

1.3 The signed equilibrium and Electrostatics

Important part of our analysis is the signed $s$-equilibrium associated with the external field $Q$ (see [6], [7]), namely a signed measure $\nu_s$, such that $\nu_s(F) = 1$ and

$$U^\nu_{\nu_s}(z) + Q(z) = C \quad \text{q.e. on } F,$$

(1.21)

where the normalization $\nu_s(F) = 1$ is chosen for convenience. If the signed equilibrium exists, then it is unique (see [7, Lemma 23]).

For logarithmic potentials finding the signed equilibrium $\nu$ may be reduced to solving a singular integral equation, namely if $d\nu = \phi(x)dx$, then by differentiating (1.21) we obtain that $\phi(x)$ is a solution to

$$\text{P.V.} \int_{F} \frac{\phi(t)}{x - t} dt = Q'(x), \quad x \in F,$$

(1.22)

where the integral is a principal value integral. The singular integral equation (1.22) is well studied in the literature (see [13], [19]). When $F$ consists of $n$ smooth arcs with ends $\{a_j\}$'s and $\{b_j\}$'s respectively and $Q'(x)$ is $\epsilon$-Hölder continuous, $0 < \epsilon < 1$, the general solution of (1.21) is given by the formula (see [19, Equation (88.9)], [13, Equation (42.29)])

$$\phi(x) = \frac{1}{\pi \sqrt{R(x)}} \left\{ P_{n-1}(x) + \frac{1}{\pi} \text{P.V.} \int_{F} \frac{\sqrt{R(t)}Q'(t)}{t - x} dt \right\}. \tag{1.23}$$

Here $P_{n-1}(z)$ is an arbitrary polynomial of degree less or equal to $n - 1$,

$$R(z) = \prod_{j=1}^{n} (z - a_j)(z - b_j),$$

and $\sqrt{R(z)}$ refers to this branch that is holomorphic in $\mathbb{C} \setminus F$. The equality (1.21) holds with the same constant for all $n$ arcs. This gives $n$ equations. With the normalization condition $\nu(F) = 1$ we have $n + 1$ equations that determine $P_{n-1}(z)$ and the equilibrium constant $C$ uniquely. This is how one can derive the formula (1.7) with the conditions (1.8) (observe that the polynomial is monic, which corresponds to the normalization condition).

For details on the corresponding hypersingular integral equation for Riesz kernels we refer to [23].

Below we explain heuristically the relationship between the signed $s$-equilibrium and the $s$-extremal measure using an electrostatics interpretation. In Electrostatics it is the signed equilibrium with normalization $\nu_s(F) = 0$ that is studied extensively, namely if we have an isolated conductor $F$ and an external field $Q(x)$,
then the electrostatic distribution of charge (signed measure) is provided by the signed equilibrium. To achieve a non-zero normalization, say \( \nu_s(F) = q > 0 \), physicists introduce a negative charge with total mass \( q \) to a grounded conductor \( F \) and then insulate the conductor. Then they remove the negative charge, resulting in positively charged conductor with total charge \( q \). Introducing external field \( Q(x) \) the electrostatics problem becomes that of finding a signed equilibrium with the normalization \( \nu_s(F) = q \) (see [14, Chapter 2] for a spherical conductor and Newton kernel). However, in the mathematical model of our set up, we look for a positive charge distribution (positive measure, not signed measure) that minimizes the potential energy. This is a considerably more difficult problem. Interestingly, it turns out that it is intricately connected to the signed equilibrium problem, as the following Lemma implies.

**Lemma 1.5** Let \( 0 \leq s < 1 \) and \( \nu_s = \nu_s^+ - \nu_s^- \) be the Jordan decomposition of the signed equilibrium \( \nu_s \) associated with the external field \( Q(x) \). Suppose further that \( U^{\nu_s} (x) \) is continuous and \( U^{\nu_s^+} (x) \) is bounded on \( F \). Then

\[
\text{supp}(\mu_{s,Q}) \subseteq \text{supp}(\nu_s^+), \quad \text{and} \quad \mu_{s,Q} \leq \nu_s^+.
\]

(1.24)

**Remark 1.6** For \( s = 0 \) this is the content of [15, Lemma 3]. For \( 0 < s < 1 \) this has been implicitly derived in the proof of Theorem 1.10 [12, p. 155-156] for the particular choice of \( F \) and \( Q \) there, but the argument can be repeated for the case here as well.

The process of finding the positive \( s \)-extremal measure from the signed equilibrium proceeds now as follows. Starting with \( F = F_0 \) we denote the signed equilibrium on \( F_0 \) associated (if it exists) with \( Q(x) \) by \( \nu_0 \). Let \( F_1 := \text{supp}(\nu_0^+) \). Since \( \text{supp}(\mu_{s,Q}) \subseteq F_1 \), we could consider the external field problem (1.11) on \( F_1 \) instead. Similarly, if we are able to find the signed equilibrium \( \nu_1 \) on \( F_1 \), we could reduce the external field problem to the set \( F_2 := \text{supp}(\nu_1^+) \). Proceeding by induction, we obtain the nested sequence of closed sets \( \{ F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots \} \), and associated signed equilibria \( \{ \nu_k \} \), such that \( \nu_0^+ \geq \nu_1^+ \geq \nu_2^+ \geq \ldots \). If we are able to show that \( \{ F_k \} \) consist of bounded number of intervals and \( \| \nu_k^- \| \to 0 \), then we would obtain \( \mu_{s,Q} = \lim \nu_k^+ \), where the convergence is weak*. This would allow us to conclude important facts for \( \mu_{s,Q} \) from facts about \( \nu_k^+ \). For example, in [8], [9], [12], [15] the nature of the equilibrium support \( \text{supp}(\mu_{s,Q}) \) was establish from information about \( \nu_k^+ \).

In the above heuristic process there are three tasks at hand. The first is to find the signed equilibria \( \{ \nu_k \} \), the second is to prove the corresponding facts we want for the sequence at hand, and the third is to show the convergence of sequence \( \{ \nu_k^+ \} \). The signed equilibria will be found using balayage techniques, introduced in the next subsection. For the second step, the main property we will need is that balayage preserves convexity of measures, in the proof of which we will utilize a technique we call ping-pong balayage. As a result, we will derive the convexity of equilibrium measures listed in Theorem 1.1. Finally, for our applications to external field problems and constrained energy problems we will derive properties for \( \mu_{s,Q} \) and \( \text{supp}(\mu_{s,Q}) \) by making use of the Iterated Balayage Algorithm (IBA) introduced in [15] for logarithmic potentials (see also [4], [8], [9]) and implicitly used in [12] for Riesz potentials.

### 1.4 Ping pong balayage of measures

An essential tool in our approach is the balayage of a measure. Given a closed set \( F \) in the complex plane \( \mathbb{C} \) and a measure \( \nu \) we call the measure \( \nu_s := \text{Bal}_s(\nu, F) \) a Riesz \( s \)-balayage of \( \nu \) onto \( F \) if \( \text{supp}(\nu_s) \subset F \) and

\[
U_s^\nu(z) = U_s^\nu_s(z) \quad \text{q.e. on } F, \quad U_s^\nu_s(z) \leq U_s^\nu(z) \quad \text{on } \mathbb{C}.
\]

(1.25)

In the logarithmic case \( s = 0 \) (per our convention we omit the subscript) we require that the logarithmic balayage \( \nu \) preserve the total mass, namely \( \| \nu \| = \| \nu_s \| \), but we relax (1.25) to

\[
U^\nu(z) = U^\nu_s(z) + c \quad \text{q.e. on } F, \quad U^\nu_s(z) \leq U^\nu(z) + c \quad \text{on } \mathbb{C},
\]

(1.26)

for some constant \( c \). Equations (1.25) and (1.26) determine the balayage measures uniquely, provided \( \text{cap}(F) > 0 \). If the set \( F \) is regular w.r.t. the Dirichlet problem, then the equalities hold everywhere (not
just quasi-everywhere). We should remark that while the logarithmic balayage preserves the total mass of a measure, Riesz $s$-balayage may decrease it. The following theorem extends the log-convexity to balayage measures.

**Theorem 1.7** Let $\nu$ be a positive measure supported on the real line/unit circle, and let $\nu(E) = 0$, where $E$ is given by (1.6). Then for any $0 \leq s < 1$, the Riesz $s$-balayage measure $\text{Bal}_s(\nu, E)$ is absolutely continuous with respect to the Lebesgue measure and its density is strictly log-convex on every subinterval/subarc of $E$.

**Remark 1.8** Let $A$ be a closed set and $\delta_t$ be the Dirac-delta measure that places unit point mass at $t \notin A$. Then, by the superposition principle,

$$d\text{Bal}_s(\nu, A) = \left( \int_{A^c} \frac{d\text{Bal}_s(\delta_t, A)}{d\nu(t)} d\nu(t) \right) du,$$

and the theorem would follow immediately once established for Dirac-delta measures.

To derive this theorem, we shall prove it directly for the case of one interval/one arc (we consider $(-\infty, a] \cup [b, \infty)$ as one interval $[b, a]$ on the real line connected at the point at infinity) and extending it to $n$ intervals/arcss utilizing a ping-pong balayage technique, the essence of which we explain below (for simplicity we illustrate it here with only two sets, although later we apply it for $n$ intervals/arcs and $n$ gaps).

Suppose we have two closed sets $F$ and $K$ in $\mathbb{C}$ (not necessarily bounded), with $\text{cap}_s(F \cap K) > 0$. For a measure $\mu$ our goal is to present $\text{Bal}_s(\mu, F \cap K)$ using balayages onto $F$ and $K$ only. With $\mu_0 := \mu$ we define recursively for $k = 0, 1, 2, \ldots$,

$$\lambda_{2k+1} := \mu_{2k+1} \cap K, \quad \mu_{2k+1} := \text{Bal}_s(\mu_{2k} - \lambda_{2k+1}, F), \quad \lambda_{2k+2} := \mu_{2k+1} \cap \overline{K}, \quad \mu_{2k+2} := \text{Bal}_s(\mu_{2k+1} - \lambda_{2k+2}, K).$$

From the balayage process it is clear that each time we balayage out of $F$ or $K$ only the part that is not located into $F \cap K$. So, heuristically we expect that

$$\text{Bal}_s(\mu, F \cap K) = \lambda_1 + \lambda_2 + \lambda_3 + \ldots,$$

and if we have proper control on the part that is left out of $F \cap K$ we will be able to prove the convergence. Since we “sweep out” consecutively the portions of the current balayage measure that are in the sets (gaps) $F \setminus K$ and $K \setminus F$, we call the technique ping-pong balayage. We can think of the mass moving to and fro the two gaps like a ping-pong ball. We expect that the measure getting trapped on the intersection in the process converges to the balayage.

For our set $E$ of $n$ compact intervals/arcs given in (1.6), we call the $n$ connected components of the complement (relative to $\mathbb{R}$ or $\mathbb{T}$) gaps. First we choose a gap $(b_j, a_{j+1})$ and balayage the part of $\nu$ sitting in this gap out of the gap. That is, we consider $\text{Bal}_s(\nu, [a_{j+1}, b_j])$ (where $a_{n+1} := a_1$). From direct calculation we know that $\text{Bal}_s(\delta_t, [a_{j+1}, b_j])$ has convex density, therefore, by Remark 1.8, $\text{Bal}_s(\nu, [a_{j+1}, b_j])$ has convex density, too.

Then going around, visiting all gaps once, in $n$ steps “sweep out” the mass of the measure that is located in the gaps, from each gap to its complement, which will bring to each subinterval of $E$ new convex contributions. Proceeding like that, we will represent $d\text{Bal}_s(\nu, E)/dt$ as a convergent series of convex functions, which will be convex. The theorem is rigorously proven in Section 4. In that proof, instead of going around, we always choose the gap which has the largest mass in it. Furthermore, an alternative inductive approach, based on a formal “ping-pong balayage” theorem will be provided in the Appendix.

Next we present the application of this theorem to the external field problem.

**Theorem 1.9** Suppose that $K$ is a compact set on the real line and that $Q(x)$ is an admissible external field and let $0 \leq s < 1$. Assume that a signed $s$-equilibrium $\nu_s$ associated with $Q$ exists and its positive part $\nu^+_s$ in its Jordan decomposition $\nu_s = \nu^+_s - \nu^-_s$ has a support that is a union of $N$ intervals having a concave density there. Then $\mu_{s,Q}$ also has a concave density and its support consists of at most $N$ subintervals, with no more than one subinterval located in an interval from the support of $\nu^+_s$. 
Theorem 1.12 Suppose that \( E \) is given by (1.6) and \( \sigma \) is a constraint measure with \( \text{supp}(\sigma) = E \), that has concave density on every subinterval of \( E \). Then for \( 0 \leq s < 1 \) the constrained s-equilibrium measure \( \lambda^s \) associated with \( \sigma \) (and no external field) has at most \( s \) subarcs where it is "free" of the constraint, and on which its density is convex. Every interval of \( E \) contains no more than one such "free subinterval".

This theorem is true for the unit circle as well (compare with Theorem 1.10).

Example 1.13 (Semi-circle constraints) Consider the logarithmic case \( s = 0 \) with

\[
E = [-3, -1] \cup [1, 3], \quad d\sigma := \frac{2}{\pi} \sqrt{|x| - 1}(3 - |x|) \, dx, \quad 1 \leq |x| \leq 3.
\]

Clearly, \( d\sigma/dx \) is concave on every subinterval of \( E \) and \( \sigma(E) = 2 \). The assumptions of the above theorem are true and from symmetry and the uniqueness of \( \lambda^s \) we may conclude that \( \text{supp}(\sigma - \lambda^s) = [-b, -a] \cup [a, b] \). Moreover, from (1.20) we have that \( \sigma - \lambda^s = \mu_Q \), where \( Q(x) = -U^s(x) \) on \( E \). Recall that \( \mu_K \) denotes the (logarithmic) equilibrium measure of \( K \). Then the endpoints \( a, b \) can be found by maximizing the Mhaskar-Saff functional (see [22, Theorem IV.1.5])

\[
F(K) = F(a, b) = \log \text{cap}(K) - \int_K Q d\mu_K
\]

over all compact sets \( K = [-b, -a] \cup [a, b] \). From [21, Corollary 5.2.6] we have

\[
\text{cap}(K) = \text{cap}([-b, -a] \cup [a, b]) = \frac{\sqrt{b^2 - a^2}}{2}, \quad 1 \leq a < b \leq 3.
\]

Furthermore,

\[
-\int_K Q(x) \, d\mu_K(x) = \int_K U^s(x) \, d\mu_K(x) = \int_E U^{\mu_K}(x) \, d\sigma(x).
\]

Using the fact that \( U^{\mu_K}(x) = -\log \text{cap}(K) - g_K(z, \infty) \), where \( g_K(z, \infty) \) denotes the Green function of \( \mathbb{C} \setminus K \), and (1.30) we derive (recall that \( \sigma(E) = 2 \))

\[
F(a, b) = \log \frac{2}{\sqrt{b^2 - a^2}} - \int_E g_K(x, \infty) \, d\sigma(x).
\]
To determine the Green function we observe that \( q(z) = z^2 \) transforms \( \mathbb{C} \setminus K \) to \( \mathbb{C} \setminus [a^2, b^2] \). Using the relation between the Green functions of the two domains (see [21, p. 134]), we obtain that
\[
g_K(z, \infty) = \frac{1}{2} g_{[a^2, b^2]}(z^2, \infty) = \frac{1}{2} \log \left| \frac{2z^2 - a^2 - b^2 + 2\sqrt{(z^2 - a^2)(z^2 - b^2)}}{b^2 - a^2} \right|. \tag{1.32}
\]
From (1.29), (1.31), and (1.32), we conclude that the endpoints \( a, b \) maximize the expression
\[
F(a, b) = \log \frac{2}{\sqrt{b^2 - a^2}} - \frac{2}{\pi} \left( \int_1^a + \int_b^3 \right) \log \left| \frac{2x^2 - a^2 - b^2 + 2\sqrt{(x^2 - a^2)(x^2 - b^2)}}{b^2 - a^2} \right| \sqrt{(x - 1)(3 - x)} dx.
\]

2 Computational preliminaries

In this section we collect the all computational results we need in the proofs of the theorems.

2.1 Riesz equilibrium measure for an interval

Let \( \mu_s \) denote the Riesz \( s \)-equilibrium measure on the interval \((b, c)\). The following formula can be essentially found in [16, Appendix 1] (see also [18, Section 2])
\[
d\mu_s = C_s \frac{dx}{(|x - b||x - c|)^{\frac{1}{2-s}}} \quad x \in [b, c], \tag{2.1}
\]
where
\[
C_s = C_{s,[b,c]} = \left[ (c - b)^s B \left( \frac{1 + s}{2}, \frac{1 - s}{2} \right) \right]^{-1}.
\]
Here \( B(x, y) \) denotes the Beta function
\[
B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt = \int_0^\infty \frac{u^{x-1}}{(1 + u)^{x+y}} du.
\]

For the sake of completeness we present a proof of this fact. Let \( y \in (b, c) \) be fixed. To compute the potential of the measure \( \mu_s \)
\[
U^{\mu_s}(y) = C_s \int_b^c \frac{1}{|x - y|^s \left( |x - b||x - c| \right)^{\frac{s}{2}}} \frac{dx}{|x - y|} = C_s \int_b^c \left( \frac{|x - y|^2}{|x - b||x - c|} \right)^{\frac{1-s}{2}} \frac{dx}{|x - y|} \tag{2.3}
\]
we use imaginary inversion change of variables
\[
x^* := \frac{|y - b||y - c|}{y - x} + y.
\]
Observe, that the points \( x \) and \( x^* \) lie on different sides of \( y \) and that \( |x - y||x^* - y| = |y - b||y - c| \). Moreover, the following formulas hold
\[
\frac{dx}{|x - y|} = \frac{dx^*}{|x^* - y|}, \quad \frac{|x - y|^2}{|x - b||x - c|} = \frac{|y - b||y - c|}{|x^* - b||x^* - c|}.
\]
Substituting in (2.3) we get
\[
U^{\mu_s}(y) = C_s \left( \int_{-\infty}^b + \int_c^\infty \right) \left( \frac{|y - b||y - c|}{|x^* - b||x^* - c|} \right)^{\frac{1-s}{2}} \frac{dx^*}{|x^* - y|} = C_s \int_c^\infty \left( \frac{|y - b||y - c|}{|x^* - b||x^* - c|} \right)^{\frac{1-s}{2}} \left( \frac{1}{x^* - y} + \frac{1}{x^* + y - b - c} \right) dx^* = C_s \int_c^\infty \left( \frac{|y - b||y - c|}{|x^* - b||x^* - c|} \right)^{\frac{1-s}{2}} \left( \frac{2x^* - b - c}{|x^* - b||x^* - c| + |y - b||y - c|} \right) dx^*. \tag{2.4}
\]
A change of variable $u := (x^* - b)(x^*- c)/(y - b)(c - y)$ in (2.4) yields
\[
U^{\mu_s}(y) = C_s \int_0^{\infty} \frac{u^{(s-1)/2}}{u + 1} du = C_s B \left( \frac{1 + s}{2}, \frac{1 - s}{2} \right) = \frac{\Gamma \left( \frac{1-s}{2} \right) \Gamma \left( 1 + s \right)}{(c - b)^s \Gamma \left( 1 + \frac{s}{2} \right)} =: \omega_s,
\]
where $\Gamma(x)$ is the Euler gamma function. Since the potential of $\mu_s$ is constant on $[b, c]$, this proves our claim. Observe that the $\mu_s$ has log-convex density.

### 2.2 Riesz balayage onto $(b, c)^c$ and $[b, c]$

Let $t \in (b, c)$ and $y \in \Sigma := ( -\infty, b ) \cup (c, \infty)$. We will determine the $s$-balayage $\delta_{s,t} := \text{Bal}_s(\delta_t, \Sigma)$. Let $b^* := 1/(b - t) + t$, $c^* := 1/(c - t) + t$, and $y^* := 1/(y - t) + t$ be the images of $b$, $c$, and $y$ under the inversion $u^* = K(u)$ with center $t$ and radius one. Let $\mu_s^*$ be the Riesz equilibrium measure for $[b^*, c^*]$. Then from (2.5)
\[
\omega_s = U^{\mu_s^*}(y^*) = C_s \int_{b^*}^{c^*} \frac{1}{|x^* - y^*|^s (|x^* - b^*||x^* - c^*|)^{\frac{1-s}{2}}} \, dx^*
\]
Applying the change of variables $x = K(x^*)$, the integral in (2.6) becomes
\[
B \left( \frac{1 + s}{2}, \frac{1 - s}{2} \right) = \int_{b^*}^{c^*} \frac{1}{|x^* - y^*|^s (|x^* - b^*||x^* - c^*|)^{\frac{1-s}{2}}} \, dx^*
\]
or equivalently
\[
\frac{1}{|t - y^*|^s} = \frac{\sin(\pi(1-s)/2)}{\pi} \int_{b^*}^{c^*} \frac{1}{|x^* - y^*|^s (|x^* - b^*||x^* - c^*|)^{\frac{1-s}{2}}} \, dx^*, \quad y \in \Sigma
\]
Here we used the formula $B(x, 1-x) = \pi/\sin(\pi x)$. Hence,
\[
d\delta_{s,t}(x) = \gamma_s \left( \frac{|t - b||t - c|}{|x - b||x - c|} \right)^{\frac{1-s}{2}} \, dx, \quad x \in \Sigma,
\]
with
\[
\gamma_s := B \left( \frac{1 + s}{2}, \frac{1 - s}{2} \right)^{-1} = \frac{\sin(\pi(1-s)/2)}{\pi}.
\]
For the general Riesz balayage of a measure $\nu$ onto $\Sigma$ we get
\[
\text{Bal}_s(\nu, \Sigma) := d\nu = d\nu|_{\Sigma} + \left( \int_{\Sigma^c} \frac{\delta_{s,t}}{d\nu(t)} \, du \right) du.
\]
In the case when $\Sigma = [b, c]$ and $t \notin [b, c]$ the considerations are modified slightly. We use the same inversion $u^* = K(u)$ with center $t$ and radius one and derive formula (2.7), which implies that (2.8) and (2.10) hold as well.

**Remark 2.14** In the logarithmic case in a similar fashion we derive the formula (for the $\Sigma = [a, b]$ case see [22, Section 2.4, Eq. (4.47)])
\[
d\nu_{0,t} = d\delta_t = \frac{\sqrt{|t - b||t - c|}}{\pi \sqrt{|u - b||u - c|} |u - t|} \, du.
\]
Observe that according to (2.11) and (2.8) both the density of the logarithmic balayage and the Riesz balayage of the Dirac-delta measure $\delta_t$ onto $\Sigma$ are log-convex and by the superposition formula (2.10) (true for $s = 0$ as well) the same can be inferred for any measure $\nu$ supported on $\Sigma^c$. Finally, we point out that $\delta_{s,t} \to \delta_t$ as $s \to 0$ and the convergence holds in terms of densities.
2.3 Riesz balayage norms

Unlike the logarithmic balayage, which preserves the norm of the measures, the Riesz balayage may reduce it. Although not crucial in proving our main results, we compute \(\|\hat{\delta}_{s,t}\|\). In the case of a balayage out of a set the norm is preserved. Indeed, if \(t \in [b, c]\), (2.4), (2.5), and (2.8) imply that

\[
\|\hat{\delta}_{s,t}\| = \gamma_s \left( \int_{-\infty}^{b} + \int_{c}^{\infty} \right) \left( \frac{|t-b||t-c|}{|x-b||x-c|} \right)^{\frac{1+s}{2}} \frac{dx}{|x-t|} = \gamma_s B \left( \frac{1+s}{2}, \frac{1-s}{2} \right) = 1
\]

(2.12)

For a general measure \(\nu\) using (2.10) one obtains

\[
\|\text{Bal}_{s}(\nu, \Omega)\| = \int_{\Omega} d\nu_{\Omega}(u) + \int_{\Omega'} \left( \int_{\Omega} \hat{\delta}_{s,t} d\nu(t) \right) du = \nu(\Omega) + \int_{\Omega'} \|\hat{\delta}_{s,t}\| d\nu(t) = \nu(\Omega) + \nu(\Omega^c) = \|\nu\|
\]

(2.13)

However, when we compute \(\|\hat{\delta}_{s,t}\|\) for the case when \(\Sigma = [b, c]\) and \(t \notin [b, c]\), say \(t < b\), we obtain after inversion about \(t\), namely \(x^* = 1/(x - t) + t\) that

\[
\|\hat{\delta}_{s,t}\| = \gamma_s \int_{b}^{c} \left( \frac{|t-b||t-c|}{|x-b||x-c|} \right)^{\frac{1+s}{2}} \frac{dx}{|x-t|} = \gamma_s \int_{c}^{b^*} \frac{1}{|x^*-t^*| ((|x^*-b^*||x^*-c^*)^{\frac{1+s}{2}}) = \frac{U^s_{\nu^*}(t)}{U^s_{\nu^*}(b^*)} < 1,
\]

(2.14)

where \(U^s_{\nu^*}(y)\) is the \(s\)-equilibrium potential on \([c^*, b^*]\).

2.4 Point balayage and equilibrium on the unit circle

In [12, Eq. (3.12)] the formula for the Riesz \(s\)-balayage of a point mass on the unit sphere \(S^d\) onto a spherical cap (for \(d - 2 < s < d\)) is given. Let \(\Sigma_0\) be the spherical cap with altitude \(t_0\), namely using cylindrical coordinates \(x = (\sqrt{1-u^2} \bar{x}, u), \bar{x} \in S^{d-1}\), for a given \(-1 < t_0 < 1\) let

\[
\Sigma_0 := \{ x = (\sqrt{1-u^2} \bar{x}, u) : u \leq t_0 \}.
\]

Suppose that \(y = (\sqrt{1-t^2} \bar{y}, t)\) is a fixed point on \(S^d \setminus \Sigma_0\). Let \(\epsilon_{\chi} = \text{Bal}(\delta_{\chi}, \Sigma_0)\) be the Riesz balayage measure. Then

\[
d\epsilon_{\chi} = A_{s,d} \left. \left( \frac{t - t_0}{t_0 - u} \right) \frac{d\sigma_d(x)}{|x-y|} \right|_{x = \chi},
\]

(2.15)

\[
A_{s,d} = \frac{\Gamma(d/2) \sin(\pi(d-s)/2)}{\pi^{d/2+1}}.
\]

(2.16)

For the case of the unit circle \(\mathbb{T} = S^1\) the formulas simplify as follows (points on \(\mathbb{T}\) are denoted by their angles). Given the closed arc \(\gamma := [\alpha, \beta]\) and a point \(\psi\) outside the arc \(\gamma\), we have that the balayage measure
Lemma 3.15 Let \( \epsilon_\psi := \text{Bal}(\delta_\psi, \gamma) \) be given by

\[
\epsilon_\psi := \frac{\sin \left( \frac{\pi(1-s)}{2} \right)}{\pi} \left( \frac{\sin \frac{\psi-\alpha}{2} \sin \frac{\psi-\beta}{2}}{\sin \frac{\theta-\alpha}{2} \sin \frac{\theta-\beta}{2}} \right)^{\frac{1-s}{2}} \frac{d\theta}{2 \sin \frac{\psi-\theta}{2}},
\]

which has clearly log-convex density in \( \theta \).

By integrating \( \epsilon_\psi \) over \( \mathbb{T} \setminus \gamma \) we show that the equilibrium measure \( \mu_{s,\gamma} \) of the arc \( \gamma \) also has a log-convex density. Indeed,

\[
\mu_{s,\gamma} = c_s \text{Bal}(d\theta, \gamma) = c_s \left( 1 + \int_{\gamma} \frac{d\epsilon_\psi}{d\theta} d\psi \right) d\theta,\gamma,
\]

where \( c_s \) is a positive normalization constant. The density of \( \mu_{s,\gamma} \) for the case of a spherical cap is found in [7, Lemma 24].

3 Mass control lemmas

In this section we provide explicit estimates that guarantee the geometric decay of mass in the gaps in the ping pong balayage process. (The mere existence of constants \( \beta_s < 1, l_s < 1 \) in the lemmas below is not difficult to derive). We first show the following lemma.

**Lemma 3.15** Let \( a < b < c < d, \, 0 \leq s < 1, \) and \( \mu \) be a measure supported on \( [b, c] \). Then

\[
\text{Bal}_s(\mu, [-\infty, b] \cup [c, \infty])([-\infty, a] \cup [d, \infty]) \leq \beta_s \|\mu\|,
\]

where

\[
\beta_s = \max \left[ \left( \frac{c-b}{b+c-2a} \right)^{1-s}, \left( \frac{c-b}{2d-b-c} \right)^{1-s} \right] < 1. \tag{3.2}
\]

**Proof:** Let

\[
\bar{\mu}_s := \text{Bal}_s(\mu, [-\infty, b] \cup [c, \infty])|_{[-\infty, a] \cup [d, \infty]}
\]

Using the formulas for balayage this means

\[
\|\bar{\mu}_s\| = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma_s \left( \frac{|t-b||t-c|}{|u-b||u-c|} \right)^{1-s} \frac{d\mu(t)}{|u-t|} du,
\]

where

\[
\gamma_s := \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \gamma_s \left( \frac{|t-b||t-c|}{|u-b||u-c|} \right)^{1-s} \frac{d\mu(t)}{|u-t|} du \right)^{1-s/2} \|\mu\|.
\]

In the last equality we used that Riesz balayage out of a set preserve norms (see (2.12))

Next we estimate

\[
\left( \frac{|t-b||t-c|}{|t-a||t-d|}, \frac{|u-a||u-d|}{|u-b||u-c|} \right)^{1-s/2} t \in [b, c], u \in [-\infty, a] \cup [d, \infty],
\]

\[
\|
\]
Using a linear transformation $w = 2(z - b)/(c - b) - 1$ we may assume $b = -1, c = 1$. Without loss of
generality we may also assume that $|a| \leq d$. Let $f(z) := (z^2 - 1)/(z - a)(z - d)$. Then we have to find

$$\max_{t \in [-1,1], u \in [a,d]} |f(t)/f(u)|.$$

The function $f(z)$ has two local extrema

$$z_1 = \frac{1 + ad + \sqrt{(a^2 - 1)(d^2 - 1)}}{a + d},$$

where $z_1 \in [-1,0]$ and $z_2 = 1/z_1$. Note that $z_2 < a$, which implies that $z_1 \in [1/a,0]$. The maximum we
search for is given by $|f(z_1)/f(z_2)|$. Hence, the maximum of the function $g(t) = |f(t)/f(1/t)|$ for $t \in [1/a,0]$
coincides with $|f(z_1)/f(z_2)|$. Since

$$g(t) = \left(\frac{1 - at}{t - a}\right) \left(\frac{1 - dt}{d - t}\right) = \left(t + \frac{1 - t^2}{t - a}\right) \left(-t + \frac{1 - t^2}{d - t}\right),$$

we see that $g(t)$ is decreasing in $d$ for any fixed $t$ and $a$. Therefore, the desired maximum is largest if $d = |a|$, in which case it is $1/a^2 < 1$. The inverse linear transform shows that $\beta_s$ is indeed given by (3.2). Using (3.3)
we now derive (3.1).

Similar estimate holds true when we balayage from the outside in. However, because we have a loss of
mass in this case (if $0 < s < 1$), for our application we need to correct the balayage measure by adding a
multiple of the equilibrium measure, so that the resulting measure has the same total mass $\|\mu\|$. Thus, given
$a < b < c < d$ and a measure $\mu$ supported on $[-\infty, a] \cup [d, +\infty]$, we denote the balayage $\mu_s := \text{Bal}_s(\mu, [a, d])$. If $\mu_s, [a, d]$ designates the $s$-equilibrium measure on $[a, d]$ (see (2.1)), the corrected measure is defined as

$$\sigma_s := \mu_s + (\|\mu\| - \|\mu_s\|)\mu_s, [a, d]$$

(3.4)

Then

$$U^{\sigma_s}(x) = U^\mu(x) + (\|\mu\| - \|\mu_s\|)\omega_s,$$

(3.5)

where $\omega_s$ is given in (2.5).

**Lemma 3.16** Let $a < b < c < d$, $0 < s < 1$, and $\mu$ be a measure supported on $[-\infty, a] \cup [d, +\infty]$. Then the measure in (3.4) satisfies

$$\sigma_s([b, c]) \leq \beta_s\|\mu\|,$$

(3.6)

where $\beta_s$ is given by (3.2).

**Proof:** Using the formulas for balayage we obtain

$$\begin{align*}
\tilde{\mu}_s([b, c]) &= \int_0^c \left[ \int_{-\infty}^a + \int_d^\infty \right] \gamma_s \left[ \frac{|u - a||u - d|}{|t - a||t - d|} \right]^{1/s} d\mu(u) dt \\
&= \int_0^c \left[ \int_{-\infty}^a + \int_d^\infty \right] \gamma_s \left[ \frac{|t - b||t - c|}{|t - a||t - d|} \frac{|u - a||u - d|}{|u - b||u - c|} \right]^{1/s} \frac{d\mu(u)}{|t - b||t - d|} dt \\
&\leq \max_{t \in [b, c], u \in [-\infty, a] \cup [d, \infty]} \left[ \frac{|t - b||t - c|}{|t - a||t - d|} \frac{|u - a||u - d|}{|u - b||u - c|} \right]^{1/s} \|\text{Bal}_s(\mu, [b, c]) \leq \beta_s\|\mu_s\|.
\end{align*}$$

(3.7)

In the last inequality we used (3.2) and the fact that $\text{Bal}(\mu, [b, c]) = \text{Bal}(\mu_s, [b, c])$ and Riesz balayage into a
set decreases the norms (see (2.14))
On the other hand
\[ \mu_{s,[a,d]}([b,c]) = C_{s,[a,d]} \int_{b}^{c} \frac{dx}{(|x-a||x-d|)^{1/s}} \]
\[ \leq \left( \frac{b-c}{d-a} \right)^{s} \max_{x \in [b,c]} \left[ \frac{|x-b|x-c|}{|x-a||x-d|} \right] \frac{1-\frac{1}{s}}{ \frac{1}{s}} C_{s,[b,c]} \int_{b}^{c} \frac{dx}{(|x-b||x-c|)^{1/s}} \]
\[ = \left( \frac{b-c}{d-a} \right)^{s} \max_{x \in [b,c]} \left[ \frac{|x-b|x-c|}{|x-a||x-d|} \right] \frac{1-\frac{1}{s}}{ \frac{1}{s}} < \beta_s, \] (3.8)
where the bound for the maximum expression can be derived as follows. We observe that if \( d-c > b-a \) then for any \( x \in [b,c] \) we could perform a similar perturbation as in Lemma 3.15 to decrease \( d \) so that \( d-c = b-a \) (we handle the case \( d-c < b-a \) by perturbing \( a \)). Since this is the symmetric case, it is easy to observe that the maximum is attained for \( x = (b+c)/2 \). This implies
\[ \max_{x \in [b,c]} \left[ \frac{|x-b|x-c|}{|x-a||x-d|} \right] \frac{1-\frac{1}{s}}{ \frac{1}{s}} \leq \beta_s, \]
as claimed. Substituting (3.7) and (3.9) in (3.4) we obtain (3.6).

Similarly, one can prove a mass control lemma for balayage measures on the unit circle.

**Lemma 3.17** Let \( 0 \leq a < b < c < d < 2\pi, \, 0 \leq s < 1, \) and \( \mu \) be a measure supported on \( [0,a] \cup [d,2\pi] \subset \mathbb{T} \) (see (1.6)). Then the measure in (3.4) satisfies
\[ \sigma_s([b,c]) \leq l_s \| \mu \|, \] (3.9)
where
\[ l_s = \max \left( \frac{1 + \cos \frac{d-a}{2}}{1 + \cos \frac{d+a-2b}{2}}, \frac{1 - \cos \frac{d+a-2b}{2}}{1 - \cos \frac{d-a}{2}}, \frac{1 + \cos \frac{d-a}{2}}{1 + \cos \frac{d+a-2b}{2}}, \frac{1 - \cos \frac{d+a-2b}{2}}{1 - \cos \frac{d-a}{2}} \right) < 1. \]

### 4 Proofs

In this section we present the proofs of the results. We first prove Theorem 1.7.

**Proof of Theorem 1.7:** We prove it only the real line case, the one for the circle is similar.

Let \( E = \cup [a_j,b_j] \) be the union of \( n \geq 2 \) intervals. We call the sets \( (b_1, a_2), \ldots, (b_{n-1}, a_n), (b_n, a_{n+1}) :=[-\infty, a_1) \cup (b_n, \infty] \) gaps.

Let \( \nu_0 := \nu \). We apply now the ping-pong balayage algorithm as follows. Let \( k \) be a non-negative integer. We will define \( \nu_k \) recursively. The density of \( \nu_k|_E \) is denoted by \( v_k(x) \) (so \( v_k(x) = 0, x \notin E \)). Let
\[ m_j := \left\| v_k \right\|_{(b_j, a_{j+1})} \]
\[ j = 1, 2, \ldots, n. \]
Choose a gap \( (b_i, a_{i+1}) \) for which \( m_i \) is maximal. Let
\[ \nu_{k+1} = \nu_k \big|_{(b_i,a_{i+1})} + Bal(\nu_k \big|_{(b_i,a_{i+1})},(b_i,a_{i+1})^c). \]
The balayage measure above has a log-convex density on \( (b_i, a_{i+1})^c \) (see Subsection 2.2), so
\[ v_{k+1}(x) = v_k(x) + K_k(x), \quad x \in E, \]
where \( K_k \) is log-convex on each subinterval of \( E \).
Clearly, \( \nu_{k+1} \) is a non-negative Borel measure with mass less than equal \( ||\nu_k|| \) (recall that the Riesz \( s \)-balayage may decrease the mass of the measure). Since \( E \) is regular we have
\[
U_{s}^{\nu_{k+1}}(x) = U_{s}^{\nu_{k}}(x) + c_k, \quad x \in E,
\]
where \( c_k \neq 0 \) only for the logarithmic case \( s = 0 \). By Lemma 3.15 and the choice of \( i \) we have
\[
\nu_{k+1}(E^c) = \sum_{j, j \neq i} m_j + (1 - \epsilon)m_i \leq \sum_{j=1}^{\infty} (1 - \epsilon/n) m_j = (1 - \epsilon/n)\nu_k(E^c).
\]
Here \( 0 < \epsilon < 1 \) is a constant which depends on \( a_1, b_1, \ldots, a_n, b_n \) only. It follows that \( \nu_k(E^c) \to 0 \) geometrically fast. And \( \nu_k(E) \) is a strictly monotone increasing sequence with \( \nu_k(E) \leq \nu_k(\mathbb{R}) \leq ||\nu|| \) for all \( k \). So \( \nu_k(E) \) is converging to a number in \( (0, ||\nu||] \). In fact, \( \nu_k(x) = \sum_{i=0}^{\infty} K_i(x), \quad x \in E \) and
\[
\nu_k(x) \neq \nu_\infty(x) := \sum_{i=0}^{\infty} K_i(x), \quad x \in E.
\]
We define \( \nu_\infty(x) \) to be zero for \( x \in E^c \). Clearly \( \nu_\infty(x) \) is log-convex on \( E \) and \( 0 < \int \nu_\infty \leq ||\nu|| \), so \( \nu_\infty(x) \) is finite valued inside \( E \).

Letting \( d\nu_\infty = \nu_\infty(x)dx \), we have
\[
U_{s}^{\nu_\infty}(x) = \lim_{k \to \infty} U_{s}^{\nu_k}(x), \quad x \in \text{int}(E),
\]
by \( \nu_k(E^c) \to 0 \) and by the monotone convergence theorem on \( E \). (Note that in the logarithmic case the functions \( t \mapsto -\log|t - x|\nu_k(t) \) may not be non-negative, however, they have a common lower bound on \( E \) for \( k = 0, 1, 2, \ldots \).) On the other hand, because of the properties of the balayage, we have
\[
U_{s}^{\nu_k}(x) = U_{s}^{\nu}(x) + d_k, \quad x \in E.
\]
(When \( 0 < s < 1 \), \( d_k = 0 \).) For \( x \in \text{int}(E) \) the left hand side has the limit \( U_{s}^{\nu_\infty}(x) \) and it is a finite value (because \( \nu_\infty \) is log-convex and therefore it is continuous on \( \text{int}(E) \)). It follows that \( d_k \) must be converging to a finite \( d \) value. This gives
\[
U_{s}^{\nu_\infty}(x) = U_{s}^{\nu}(x) + d, \quad x \in \text{int}(E). \tag{4.2}
\]

Although a short way to finish our proof would be to argue that \( U_{s}^{\nu_\infty}(x) \) must be bounded on \( \partial E \), we prefer to present an argument showing that \( (4.2) \) holds also for \( x \in \partial E \).

We have established \( \nu_k(E^c) \leq Dq^k, \quad k = 1, 2, \ldots, \) where \( 0 < q < 1 \). Because of \( \nu_k(E^c) \to 0 \), we must visit each gap in the process infinitely many times. Let \( j \) be fixed. When we visit the \((b_j, a_{j+1})\) gap at step \( l \), we sweep out the mass from there so \( U_{s}^{\nu_k}|_{E^c}(b_j) \) is clearly small if \( l \) was large.

On the other hand, if \((b_i, a_{i+1})\) is any gap, \( i \neq j \), and \( t \in (b_i, a_{i+1}) \), \( \mu_t := \text{Bal}(\delta_t, (b_i, a_{i+1})^c) \), then \( U_{s}^{\nu_k}|_{E^c}(b_j) \leq M \) where \( M \) depend on \( a_1, \ldots, b_n \) only.

Suppose the gap \((b_j, a_{j+1})\) was visited at step \( k_0 \), where \( k_0 \) is large enough, and then again at step \( k_1 \). Between the two visits, we visit the gaps \((b_{ik}, a_{ik+1})\), \( k = k_0 + 1, \ldots, k_1 - 1 \). For any \( l \in \{k_0 + 1, \ldots, k_1 - 1\} \) we have
\[
U_{s}^{\nu_k}|_{E^c}(b_j) \leq U_{s}^{\nu_{k_0}}|_{E^c}(b_j) + \sum_{k=k_0+1}^{k_1-1} M\nu_k((b_{ik}, a_{ik+1}))
\]
\[
\leq U_{s}^{\nu_{k_0}}|_{E^c}(b_j) + M \sum_{k=k_0+1}^{k_1-1} Dq^k \leq U_{s}^{\nu_{k_0}}|_{E^c}(b_j) + MD \sum_{k=k_0+1}^{\infty} q^k,
\]
which is small if \( k_0 \) is large. This implies \( U_{s}^{\nu_k}|_{E^c}(b_j) \to 0 \), as \( l \to \infty \). Similarly, \( U_{s}^{\nu_k}|_{E^c}(x) \to 0 \), for any \( x \in \partial E \), as \( l \to \infty \).
Thus, (4.1) also holds for \( x \in \partial E \) by the monotone convergence theorem, and
\[
U_s^{\nu_\infty}(x) = U_s^\nu(x) + d, \quad x \in E.
\] (4.3)

For the Riesz kernel \( d = 0 \) since all \( d_k = 0 \). For the logarithmic kernel we have \( ||\nu_\infty|| = ||\nu|| \) since \( ||\nu_{k+1}|| = ||\nu_k|| \) for all \( k \). Using the Principle of Domination (see [22, Theorem 3.2] for the logarithmic case and [16, Theorem 1.29] for the Riesz case) we derive the inequality
\[
U_s^{\nu_\infty}(x) \leq U_s^\nu(x) + d, \quad x \in \mathbb{C}.
\] (4.4)

In conclusion, (4.3) and (4.4) and the uniqueness of the balayage measure for regular sets imply \( \text{Bal}(\nu, E) = \nu_\infty \) and it has log-convex density on \( E \).

We now utilize Theorem 1.7 into the proof of Theorem 1.1.

**Proof of Theorem 1.1:** We shall prove the theorem simultaneously for both the real line and the circle cases. Define the measure
\[
\sigma_{s, E} := \text{Bal}(\mu_s, [a_1, b_n], E) = (\mu_s, [a_1, b_n])|E + \text{Bal}(\mu_s, [a_1, b_n])|E^c, E).
\] (4.5)

Since its potential on \( E \) satisfies
\[
U_{s, E}^{\sigma_{s, E}}(x) = U_s^{\mu_s, [a_1, b_n]}(x) + \text{const} = \text{const} \quad \text{for all} \quad x \in E,
\]
the equilibrium measure on \( E \) is given by \( \mu_{s, E} = c\sigma_{s, E} \) for some positive constant \( c \) (for Riesz kernels we will lose mass in the balayage process and we need to re-normalize). From Subsection 2.1 for the real line and Subsection 2.4 for the circle case we have that the first component of \( \sigma_{s, E} \) in (4.5) is log-convex. The second one, being a balayage measure is log-convex from Theorem 1.7. This proves the theorem.

**Proof of Theorem 1.9:** The proof is based on the iterated balayage algorithm (for two intervals see [4]). Here we give the proof for \( n \) intervals on the real line. The circle case (Theorem 1.10) is handled similarly.

Let \( \Sigma \) be a closed subset of the real line and \( Q \) be an external field defined on \( \Sigma \). Let \( \Sigma_0 := \Sigma \) and let \( \sigma_0 \) denote the signed equilibrium associated with \( Q \). By the assumption of the theorem the latter exists, and if \( \sigma_0 = \sigma_0^+ - \sigma_0^- \) is the Jordan decomposition of \( \sigma_0 \), the measure \( \sigma_0^- \) has support \( \Sigma_1 := \text{supp}(\sigma_0^+) = \bigcup_{j=1}^N [a_j^{(1)}, b_j^{(1)}] \) and \( \sigma_0^+ \) has concave density there.

Recall that \( k_s(x, y) \) stands for the logarithmic kernel \( -\log |x-y| \) or Riesz kernel \( |x-y|^{-s} \) (see (1.1)). Define recursively
\[
\sigma_{n+1} = \sigma_n^+ - [\text{Bal}(\sigma_n, \Sigma_{n+1}) + c\mu_{s, \Sigma_n}],
\] (4.6)
and \( \Sigma_{n+2} := \text{supp}(\sigma_{n+1}^+) \), \( n = 0, 1, 2, \ldots \) where \( \mu_{s, \Sigma_{n+1}} \) is the equilibrium measure of the set \( \Sigma_{n+1} \). When \( k_s(x, y) \) is the log kernel, we choose \( c = 0 \), but when it is the Riesz kernel, we may lose mass when balayaging \( \sigma_n^- \), so we choose \( 0 \leq c \) so that the measure in the bracket has the same mass as \( \sigma_n^- \). Note also that the s-equilibrium measure has convex density, so the measure in the bracket has convex density, too.

By induction we see that \( \Sigma_0 \supset \Sigma_1 \supset \ldots \), each \( \Sigma_n \) contains at most one subinterval of \( \Sigma_{n+1} \) (so we have a system of nested intervals), finally, \( \sigma_n^+ \) has convex density on \( \Sigma_{n+1} \). All these follow from \( \text{Bal}(\sigma_n^-, \Sigma_{n+1}) + c\mu_{s, \Sigma_n} \) being a convex function.

Now we show that the density of \( \sigma_n \), which we will denote by \( v_n(x) \), is converging to the density of the equilibrium measure \( \mu_{s, \Sigma} \) associated with the external field \( Q(x) \).

Note that we may “lose” an interval in the IBA if at one step \( v_n \leq 0 \) on \([a_j^{(n)}, b_j^{(n)}] \). However this loss of intervals will eventually stop as \( n \to \infty \). So without loss of generality let us assume that for all \( n \) we have \( \Sigma_n = \bigcup_{j=1}^N [a_j^{(n)}, b_j^{(n)}] \), where \([a_j^{(n)}, b_j^{(n)}] \) are \( N \) non-trivial disjoint intervals.

Let \( \lim_{n \to \infty} a_j^{(n)} = a_j \), \( \lim_{n \to \infty} b_j^{(n)} = b_j \), so \( a_j \leq b_j \) and let \( \Sigma' := \bigcup_{j=1}^N [a_j, b_j] \).

Note that \( U_s^{\sigma_k^+}(x) \) is continuous on \( \Sigma_k \), because \( \sigma_k^+ \) has concave density. Also, \( U_s^{\sigma_k^+}(x) - U_s^{\sigma_k^-}(x) + Q(x) = F_k \) on \( \Sigma_k \), so \( U_s^{\sigma_k^+}(x) \) is bounded (in fact continuous) on \( \Sigma_k, k \geq 1 \). Applying Lemma 1.5 inductively we see that \( \text{supp}(\mu_{s, \Sigma}) \subset \Sigma' \).
We have

\[ F_n - Q(x) = \int_{\Sigma_n} k_s(t, x) v_n(t) dt = \left( \int_{\Sigma_{n+1}} + \int_{\Sigma_n \setminus \Sigma_{n+1}} \right) k_s(t, x) v_n(t) dt \]  \tag{4.7} \]

We claim that

\[ \int_{\Sigma_n \setminus \Sigma_{n+1}} v_n(t) dt \to 0. \]  \tag{4.8} \]

If not, then there exist a subsequence, denoted by \( n \) for simplicity, and \( m_j \leq 0, j = 1, \ldots, N, M_j \leq 0, j = 1, \ldots, N \),

\[ \int_{[a_j^{(n)}, a_j^{(n+1)}]} v_n(t) dt \to m_j, \quad \int_{[b_j^{(n+1)}, b_j^{(n)}]} v_n(t) dt \to M_j, \quad \text{and not all } m_j \text{ and } M_j \text{ values are zero.} \]

Let \( v := \lim v_n \). Since \( v_n \big|_{\Sigma_{n+1}} \) is a bounded decreasing sequence of non-negative functions, we can use the dominated convergence theorem and mean value theorem to derive that for any \( x \in \text{int}(\Sigma') \)

\[ \lim_{n \to \infty} (F_n - Q(x)) = \int_{\Sigma'} k_s(t, x) v(t) dt + \sum_{j=1}^{N} m_j k_s(a_j, x) + \sum_{j=1}^{N} M_j k_s(b_j, x). \]  \tag{4.9} \]

All \([a_j, b_j] \) intervals cannot be trivial, so let us assume that \( a_1 < b_1 \). Letting \( x := (a_1 + b_1)/2 \) at (4.9), we gain that \( \lim_{n \to \infty} F_n \) is finite. Now we also see that \( m_j = 0, M_j = 0 \) \( (j = 1, \ldots, N) \) must hold. For example, if \( m_1 < 0 \), then (4.9) is also valid for \( x = a \) in the sense that we have negative infinity on the right-side of the equation. But this is a contradiction.

Let \( F := \lim F_n \). From (4.7) and (4.8) we get that for any \( x \in \text{int}(\Sigma') \)

\[ F - Q(x) = \int_{\Sigma'} k_s(t, x) v(t) dt. \]  \tag{4.10} \]

From (4.8) it is also clear that \( v(t) \) is a probability density function. Further, it is bounded, so \( v(t) \) \( dt \) has finite energy. Recall that the support of \( \mu_{s,Q} \) is a subset of \( \Sigma' \). The equilibrium measure \( \mu_{s,Q} \) minimizes the weighted energy on \( \Sigma \) and therefore on \( \Sigma' \), too. By (4.10) it follows that \( v(x) \) is the density of the equilibrium measure (see [22, Theorem 1.3.3] for \( s = 0 \) and [27, Theorem 1 and Proposition 1] for \( 0 < s < 1 \)).

Finally, note that \( \text{supp}(\mu_{s,Q}) \) is the union of those \([a_j, b_j] \) which are non-trivial intervals. \( \square \)

**Proof of Theorem 1.12:** By the discussion at the end of Subsection 1.2 \( \mu_{s,Q} = (\|\sigma\| - 1)^{-1}(\sigma - \lambda_s^\sigma) \) solves an external field problem with \( Q(x) = -((\|\sigma\| - 1)^{-1}U^\sigma(x) \) (see (1.20)). However, the signed equilibrium in this case is easy to obtain and is given by \( \nu_{s,Q} = (\|\sigma\| - 1)^{-1}(\sigma - \mu_{s,E}) \). Indeed,

\[ U^{\nu_{s,Q}}(x) + Q(x) = -((\|\sigma\| - 1)^{-1}U^{\mu_{s,E}}(x) = -((\|\sigma\| - 1)^{-1}W_s(E) \text{ for all } x \in E. \]

But \( \mu_{s,E} \) has convex density on \( E \) by Theorem 1.1. By the hypothesis \( \sigma \) has concave density, and so the signed equilibrium \( \nu_{s,Q} \) has a concave density on every subinterval of \( E \). By Theorem 1.9 we derive that the set \( K := \text{supp}(\mu_{s,Q}) = \text{supp}(\sigma - \lambda_s^\sigma) \) is a finite union of compact intervals contained in \( E \) and that every subinterval of \( E \) contains no more than one subinterval of \( K \).

Note that \( \lambda_s^\sigma = \sigma - (\|\sigma\| - 1)\mu_{s,Q} \). Let \( S := \text{supp}(\mu_{s,Q}) \). Recall from the proof of Theorem 1.9 that we found \( \mu_{s,Q} \) with the iterated balayage algorithm. We started with the signed equilibrium measure \( \nu_{s,Q} \) and the process resulted

\[ \mu_{s,Q} = \left[ \nu_{s,Q} - \sum_{j=1}^{\infty} \eta_j \right]_{S}. \]
where \( \nu_{s,Q} \) has concave density and all \( \eta_j \) have convex densities on \( S \) (see equation (4.6)).

Thus,

\[
\lambda_s^\nu = \sigma - (||\sigma|| - 1)\mu_{s,Q} = \sigma - (||\sigma|| - 1)\left[\nu_{s,Q} - \sum_{j=1}^{\infty} \eta_j\right]_S \\
= \sigma - (||\sigma|| - 1)\left[\left(1 - \frac{1}{||\sigma|| - 1}\right)(\sigma - \mu_{s,E}) - \sum_{j=1}^{\infty} \eta_j\right]_S \\
= \sigma|_{E\setminus S} + \left[\mu_{s,E} + (||\sigma|| - 1)\sum_{j=1}^{\infty} \eta_j\right]_S.
\]

The measures in the last bracket have convex densities on \( S \), concluding our proof. \( \square \)

5 Appendix

In this section we shall prove a general ping-pong balayage theorem for closed subsets \( F \) and \( K \) of the real line or the unit circle. For simplicity, let’s assume \( F, K \), and \( F \cap K \) are \( s \)-regular with respect to the Dirichlet problem (see [16, Section IV.5, p. 263]). We will apply the theorem to sets of type (1.6), which are \( s \)-regular.

**Theorem 5.18 (Ping pong balayage theorem)** Let \( F \) and \( K \) be closed sets of \( \mathbb{R} \) or \( \mathbb{T} \), such that \( F, K \), and \( F \cap K \) are all \( s \)-regular and let \( \mu_0 \) be a measure supported on \( K \), that has bounded \( s \)-potential on \( F \cap K \). Assume further that \( \text{dist}(F \setminus K, K \setminus F) > 0 \) and that \( F \cap K \) contains an open interval. The ping-pong balayage sequence \( \{\mu_n\} \), formed recursively as \( \mu_{2n+1} = \text{Bal}(\mu_{2n}, F) \), \( \mu_{2n+2} = \text{Bal}(\mu_{2n+1}, K) \), \( n = 0, 1, \ldots \), is weak* convergent with \( \lim \mu_n = \mu := \text{Bal}(\mu_0, F \cap K) \), i.e. \( \lim_{n \to \infty} \int f(\lambda) d\mu_n(\lambda) = \int f(\lambda) d\mu(\lambda) \) for any compactly supported continuous function \( f(\lambda) \).

**Proof:** We prove only the real line case, the circle is similar. The regularity of the sets \( F, K \), and \( F \cap K \) guarantees that all balayage measures \( \{\mu_n\} \) exist and are uniquely determined by \( \mu_0 \). Since balayage decreases the total mass of a measure, then the sequence is weak* compact. Suppose \( \lambda \) is a weak* cluster point, i.e. there is a subsequence \( \{\mu_{n_k}\} \) such that \( \mu_{n_k} \rightharpoonup \lambda \) as \( k \to \infty \). We shall prove that \( \lambda = \text{Bal}(\mu_0, F \cap K) \) and the theorem will follow because any subsequence has a subsequence which converge to \( \lambda \).

From the definition of \( s \)-balayage (see Section 1.4, (1.25) and (1.26)) we have for \( n = 1, 2, \ldots \)

\[
U^\lambda_k(x) = U^\mu_k(x), \quad \text{x \in F \cap K}, \quad U^\lambda_k(x) \leq U^\mu_k(x), \quad \text{x \in R}.
\]

The Lower Envelope Theorem ([22, Theorem I.6.9] for \( s=0 \) and [16, Theorem 3.8] for \( 0 < s < 1 \)) and the Principle of Descent ([22, Theorem I.6.8] and [16, Theorem 1.3] respectively) allow us to conclude that

\[
U^\lambda_k(x) = U^\mu_k(x), \quad \text{q.e. x \in F \cap K}, \quad U^\lambda_k(x) \leq U^\mu_k(x), \quad \text{x \in R}. \quad (5.1)
\]

Next we establish that there is an \( 0 < q < 1 \), such that

\[
\mu_{2n+1}(F \setminus K) \leq q \mu_{2n}(K \setminus F), \quad \mu_{2n+2}(K \setminus F) \leq q \mu_{2n+1}(F \setminus K), \quad n = 0, 1, \ldots \quad (5.2)
\]

The assumptions on \( F \) and \( K \) imply that there exists \( \epsilon > 0 \), such that \( F \cap K \) contains an interval of length \( 2\epsilon \) and

\[
F \setminus K \subset F_\epsilon := \left( F \cap \left\{ t - \epsilon, t + \epsilon \right\}, \ t \in K \setminus F \right), \quad K \setminus F \subset K_\epsilon := \left( K \cap \left\{ t - \epsilon, t + \epsilon \right\}, \ t \in F \setminus K \right).
\]

Let us prove the first of the inequalities in (5.2) (the other being similar). From the definition \( \text{supp}(\mu_{2n}) \subset K \). Observe that \( \mu_{2n+1}|_{F \setminus K} = \text{Bal}(\mu_{2n}|_{K \setminus F}, F) \). Let

\[
\nu_\epsilon := \text{Bal}(\mu_{2n}|_{K \setminus F}, F_\epsilon).
\]

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Clearly, $\nu_c = \mu_{2n+1}|_{F_c} + \mathrm{Bal}(\mu_{2n+1}|_{F\setminus F_c}, F_c)$, so

$$\mu_{2n+1}(F \setminus K) \leq \nu_c(F \setminus K).$$

(5.3)

On the other hand, the superposition principle (1.27) implies that

$$d\nu_c = \left( \int_{K \setminus F} \frac{d\mathrm{Bal}(\delta_t, F_c)}{du} \mu_{2n}(t) \right) du.$$

Since for any $t \in K \setminus F$ we have $F_c \subset (t - \epsilon, t + \epsilon)^c$,

$$\mathrm{Bal}(\delta_t, F_c) = \mathrm{Bal}(\delta_t, (t - \epsilon, t + \epsilon)^c)|_{F_c} + \mathrm{Bal}(\mathrm{Bal}(\delta_t, (t - \epsilon, t + \epsilon)^c)|(t - \epsilon, t + \epsilon)^c \cap F_c, F_c),$$

so the inequality

$$\mathrm{Bal}(\delta_t, F_c)(F_c \cap K) \geq \mathrm{Bal}(\delta_t, (t - \epsilon, t + \epsilon)^c)(F_c \cap K)$$

holds. From (2.8), (2.10), and (2.11) ((2.17) for the unit circle case) we can conclude that there exists a positive constant $C_\epsilon$ independent of $t \in K \setminus F$ such that $\mathrm{Bal}(\delta_t, (t - \epsilon, t + \epsilon)^c)(F_c \cap K) \geq C_\epsilon$. Indeed, because $F \cap K$ contains an interval of length $2\epsilon$, then $F_c \cap K$ contains an interval of length $\epsilon$ that is at least $\epsilon$ distance away from $K \setminus F$. This shows that $\nu_c(F_c \cap K) \geq C_\epsilon\mu_{2n}(K \setminus F)$, or equivalently

$$\nu_c(F \setminus K) \leq (1 - C_\epsilon)\mu_{2n}(K \setminus F).$$

The latter inequality together with (5.3) yields the first inequality in (5.2) with $q = 1 - C_\epsilon$. The second is proved similarly.

Returning to the weak* limit $\lambda$, (5.2) implies that $\text{supp}(\lambda) \subset F \cap K$, which together with (5.1) implies that $\lambda = \mathrm{Bal}_s(\mu_0, F \cap K)$, and thus proves the theorem. $\square$

We now provide an alternative proof of Theorems 1.1 and 1.7.

Alternative proof of Theorem 1.8: We shall only consider the point mass balayage onto finitely many intervals, i.e. let $E = \cup_{i=1}^n [a_i, b_i]$, $y \not\in E$. We want to prove that $\mathrm{Bal}_s(\delta_y, E)$ has log-convex density. We shall proceed by induction on the number of intervals $n$. For $n = 1$ this is true from Section 2.2.

If $n = 2$, we set $F = (-\infty, b_1] \cup [a_2, \infty)$ and $K = [a_1, b_2]$. Clearly, $E = F \cap K$ and with $\mu_0 = \delta_y$ we can apply Theorem 5.18 to derive the weak* convergence of the ping-pong sequence $\mu_n$. Let the density of $\mu_n$ be $H_n$ and the density of $\mathrm{Bal}_s(\delta_y, E)$ be $H(x)$. By the weak* convergence $H_n(x) \to H(x)$ pointwise $(x \in E)$.

As in the proof of Theorem 1.7 we also see that $H_n' \to H'$ inside $E$, and $H'$ is log-convex as we claimed.

For general $n \geq 3$, let us choose $F = E \cup [b_1, a_2]$ and $K = E \cup [b_2, a_3]$. Then both $F$ and $E$ will be unions of $n - 1$ intervals and by an inductive hypothesis the ping-pong sequence will have convex densities. The same argument applies. $\square$

References


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