The recent volatility of the stock market has left most investors stunned. In 1952 only 4 percent of the U.S. population owned stocks, but today about half of American households do. Despite this, most investors have no education in either economics or the stock market. They instead rely on the recommendations of their financial advisers. The adviser typically shows the client a chart of the stock market’s history, commenting that short-term volatility is normal, but stocks have been going up in the long run. The chart is very convincing, and it is easy to accept the “buy and hold” argument. The problem when looking at that chart is twofold:

- Stock prices behave very much like a random walk, and therefore future performance is independent of the past. As Warren Buffett once said: “If past history was all there was to the game, the richest people would be librarians.”

- Over the past 20 years, the stock price of Microsoft has increased more than any other. Hence, showing the chart of Microsoft as a typical example would be misleading. By the same logic, looking only at the chart of the U.S. stock market is meaningless. In the past century, the United States was one of the fastest growing economies in the world. Other countries were not so lucky, as their economies grew more slowly.

Modeling Stock Prices
The simplest way to think about a stock price is to assume that it represents the value of the company per share in 30 years. That is, \( \text{FutureValue} = N \cdot P \), where \( N \) is the number of shares outstanding and \( P \) is the current share price. We assume that in 30 years the company will go bankrupt, its assets will be sold, and stockholders will get the proceeds. (This is a realistic assumption, since most companies do go bankrupt after being in business for a few decades.) Because 30 years is a long time, even a small change in the annual growth expectation of the company will have a huge impact on its future value and, therefore, a huge impact on the current share price. For example, if the growth expectation is changing from 3 to 3.5 percent, then the share price will change from \( P \) to \( \frac{1.035}{1.03}P \), which is a jump of about 16 percent.
This explains why stock prices can move up and down so rapidly. For example, the stock price of a large steel manufacturing company was around $200 in June 2008, and four months later it had fallen to $30.

In order to gain a better understanding of the dynamics of the stock market, we will need to consider a model that incorporates randomness. In this model, \( w(t) \) will denote a Brownian motion. (See figure 1.) The reader may think of it as a random, continuous function with the property that for any fixed \( t \geq 0 \), \( w(t) \) is a random variable having normal distribution with mean zero and standard deviation \( \sqrt{t} \). In particular, \( w(0) = 0 \). (To get familiar with the concept of Brownian motion, one may think of the discrete Brownian motion: each year we flip a coin, and if it is heads we win $1; otherwise, we lose $1. If \( w(n) \) denotes the total gain during the first \( n \) years, then \( w(n) \) has mean zero and standard deviation \( \sqrt{n} \).)

Let us consider a stock that is worth $1 at time \( t = 0 \). A simple but frequently used model for the stock price is

\[ P(t) = e^{\mu t + \sigma w(t)} \]

where \( t \) is the time in years, \( \mu \) is a constant, called the drift factor, and \( \sigma \) is the volatility of the stock price. That is, \( \sigma \) is the standard deviation of the change in \( \ln P(t) \) during one year. Although this model is simple, it is very natural, and it is also used in the celebrated Black-Scholes option-pricing model.

For fixed \( t \), the random variable \( w(t) \) has density function

\[ e^{-z^2/2} / \sqrt{2\pi t}, \quad z \in \mathbb{R} \]

and one can easily verify that the expected value of \( P(t) \) is

\[ E(P(t)) = e^{(\mu + \sigma^2/2)t} = e^{Rt}, \quad \text{where} \quad R := \mu + \sigma^2/2. \]

This means that at time \( t \), our average gain is \( e^{Rt} - 1 \), since our original investment was $1.

A bank offering interest rate \( r \) compounded continuously would give us \( e^{rt} - 1 \) gain, which is (almost) risk free. The Treasury bill with three-month maturity is considered to be a risk-free bond. We will use the letter \( r \) for the interest rate of the Treasury bill. Investors wish to be compensated for taking risk when buying stocks, so \( R > r \) must hold—at least theoretically. (Because we do not know the exact value of \( R \), it may happen that \( R \leq r \), in which case the stock is overvalued.)

The difference \( R - r \) is called the equity risk premium.

**Everything Is Relative!**

If we invest $1 in a stock and the stock price drops, say, to 80 cents, it is very common for the broker to call it a “paper loss.” He claims that it is not a real loss unless we sell the stock. Nothing is further from the truth. Just think of a casino: If you lost some chips, it is a real loss, even though you did not cash out!

True, if you don’t sell and wait \( t \) years, the stock may come back to $1, and you may feel satisfied that you waited. But keep in mind that if you took your losses and put the 80 cents in Treasuries, after \( t \) years you would have 0.80\(e^{rt} \) dollars, which can be more than $1, depending on how large \( t \) is.

It is clear now that it is not enough to calculate our raw gain on an investment. The real question is whether our investment is growing faster or slower than the risk-free Treasury.

**Definition:** The value of the investment of $1 at time \( t \) relative to the risk-free Treasury bill is

\[ \text{Rel}(t) = \frac{P(t)}{e^{rt}} = e^{(\mu - r)t + \sigma w(t)}. \]

We can think of \( \text{Rel}(t) \) as our “normalized money” (see figure 2). At time \( t \) if \( \text{Rel}(t) < 1 \) we will simply say that we lost money, and if \( \text{Rel}(t) > 1 \) we will say that we gained money. Similarly, if, say, \( \text{Rel}(t) = 0.5 \) we will simply say that we lost half of our money. We will stop saying “relative to the risk-free Treasury bill” since \( e^{rt} \) is the money we deserve risk free; it must be the basis for comparison (not the money under the pillow earning no interest). The expected value of our normalized money is \( E(\text{Rel}(t)) = e^{(R - r)t} > 1 \), because we assumed that \( R > r \).

From \( R = \mu + \sigma^2/2 \), we get

\[ \text{Rel}(t) = e^{(R - r - \sigma^2/2)t + \sigma w(t)}, \]

so \( \text{Rel}(t) \) has drift factor \( R - r - \sigma^2/2 \).
More Time, Less Risk?

Let $t > 0$ be a fixed time, our investment horizon. For example, we might set $t = 20$ years. Let us find the probability that $\text{Rel}(t)$ is in the interval $(a, b)$, where $0 \leq a < b \leq \infty$.

$$P(a < \text{Rel}(t) < b) = P(a < e^{(R-r-\sigma^2/2)t+\sigma w(t)} < b) = P\left(\frac{\ln a - (R-r-\sigma^2/2)t}{\sigma \sqrt{t}} < \frac{w(t)}{\sqrt{t}} < \frac{\ln b - (R-r-\sigma^2/2)t}{\sigma \sqrt{t}}\right).$$

Here $w(t)/\sqrt{t}$ is a random variable with standard normal distribution. Its density is $\frac{e^{-x^2/2}}{\sqrt{2\pi}}$, so the probability is

$$P(a < \text{Rel}(t) < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx,$$

where the integration takes place on the interval

$$I = \left(\frac{\ln a - (R-r-\sigma^2/2)t}{\sigma \sqrt{t}}, \frac{\ln b - (R-r-\sigma^2/2)t}{\sigma \sqrt{t}}\right).$$

Further, it follows that

$$P(\text{Rel}(t) < b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^b e^{-x^2/2} dx,$$

where $I = \left(-\infty, \frac{\ln b - (R-r-\sigma^2/2)t}{\sigma \sqrt{t}}\right)$.

and

$$P(\text{Rel}(t) < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx,$$

where $I = \left(-\infty, \frac{\ln b - (R-r-\sigma^2/2)t}{\sigma \sqrt{t}}\right)$.

To apply these formulas, we need to find a proper value for the risk premium $R - r$ and the volatility $\sigma$. For Standard & Poor's 500 index (which contains 500 large companies), data are available for both the historical risk premium and volatility. In addition, diversification is important when investing. So we will assume that instead of investing in a single stock, we invest in the S&P 500 index.

In real life, $R$, $\sigma$, and even $r$ are not constant values. Recently $r$ has been extremely low: $r = 0.001$ (0.1 percent), but we cannot assume that this will be typical in the future. In the United States, from 1926 through 2005 the stock index of large companies grew with an annualized rate of $R = 0.104$ (10.4 percent) on average, and on average $r = 0.037$ was the Treasury rate (see W. N. Goetzmann and R. G. Ibbotson, *The Equity Risk Premium: Essays and Explorations*, page 35). This would give $R - r = 0.067$. But during the 1802–1925 period, the risk premium was only 2.6 percent (J. J. Siegel, *Stocks for the Long Run*, 3rd ed., page 18). In the past century, historical market returns in the world averaged 3.8 percent plus the rate of inflation (W. N. Goetzmann and R. G. Ibbotson, *The Equity Risk Premium: Essays and Explorations*, pages 343 and 348). Assuming that in each country $r$ matches the rate of inflation, we get a historical risk premium of 3.8 percent for the world.

Even among economists, there is a vigorous debate about the value of the equity premium. Their estimates vary from around zero to 6 percent, so for our calculations we will use $R - r = 0.03$.

It is also reasonable to set $\sigma = 0.20$, since this was the (annualized) standard deviation of the S&P 500 stock index over the past 10 years. (As a curiosity we mention that during October and November in 2008, the volatility was a remarkable 0.76!)

Financial advisers often claim that we should ignore the volatility of stocks because stocks beat bonds in the long run; at least historically they did during almost any 20-year period (Siegel's *Stocks for the Long Run*, page 28). Let us examine whether this claim is supported by our model.

Let $t = 20$ years. The expected value $E(\text{Rel}(20)) = e^{0.03\cdot20} = 1.82$ suggests that stocks are great investments since we will have on average almost twice as much money as we would with risk-free Treasury notes. But this is only an average. By (1),

$$P(\text{Rel}(20) < 1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^1 e^{-x^2/2} dx,$$

with $v = \frac{\ln 1 - (0.03 - 0.20^2/2)20}{0.20\sqrt{20}} = -0.2236$. 

![Figure 2. A possible outcome over 20 years.](image-url)
the probability of losing money is \( P(\text{Rel}(20) < 1) = 0.41 = 41\% \). (The integral needs to be evaluated numerically.)

We also calculated the expected value \( E(\text{Rel}(t)) \) and the probabilities \( P(\text{Rel}(t) \leq 0.5) \), \( P(\text{Rel}(t) < 1) \), and \( P(\text{Rel}(t) \geq 2) \) for the time periods \( t = 5, 20, 40, \) and 100 years (see table).

<table>
<thead>
<tr>
<th>( t ) years</th>
<th>( E(\text{Rel}(t)) )</th>
<th>( P(\text{Rel}(t) \leq 0.5) )</th>
<th>( P(\text{Rel}(t) &lt; 1) )</th>
<th>( P(\text{Rel}(t) \geq 2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1.16</td>
<td>0.05</td>
<td>0.46</td>
<td>0.08</td>
</tr>
<tr>
<td>20</td>
<td>1.82</td>
<td>0.16</td>
<td>0.41</td>
<td>0.29</td>
</tr>
<tr>
<td>40</td>
<td>3.32</td>
<td>0.19</td>
<td>0.38</td>
<td>0.41</td>
</tr>
<tr>
<td>100</td>
<td>20.09</td>
<td>0.20</td>
<td>0.31</td>
<td>0.56</td>
</tr>
</tbody>
</table>

Observe that investing for a longer time does not really save us from the risk of losing money (even though our expected value is increasing with time). In fact, it is very interesting that \( P(\text{Rel}(t) \leq 0.5) \) seems to be decreasing with \( t \)—just the opposite of what one might suspect. Actually, it increases until \( t = 69.3 \) years, and then it begins decreasing, eventually approaching zero. If we measure risk as the probability of losing half our money, financial advisers are right until \( t = 69.3 \) years, and then it begins decreasing, eventually approaching zero. If we measure risk as the probability of losing at least half our money, financial advisers are right about emphasizing long-term investing, but only if the long term is well over 69.3 years!

There is also some good news: the probability of losing money is decreasing with time, because \( R - r - \sigma^2 / 2 = 0.01 \) is positive. Indeed, we have seen that

\[
P(\text{Rel}(t) < b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{v(t)} e^{-x^2} dx,
\]

where

\[
v(t) = \frac{\ln b}{\sigma \sqrt{t}} - \frac{R - r - \sigma^2 / 2}{\sigma} \sqrt{t}.
\]

For \( b = 1 \), \( v(t) \) is strictly decreasing; hence, \( P(\text{Rel}(t) \leq 1) \) is strictly decreasing, too. (One can also analyze \( P(\text{Rel}(t) \leq b) \) for arbitrary \( b \) using \( v(t) \).)

**Can We Swim against the Drift?**

With our choices \( R - r = 0.03 \) and \( \sigma = 0.20 \), we have \( R - r - \sigma^2 / 2 = 0.01 > 0 \). But can the drift \( R - r - \sigma^2 / 2 \) be negative? Obviously if the risk premium \( R - r \) is less than 0.20^2 / 2 = 0.02, then the drift is negative. But even with \( R - r = 0.03 \), we can have a negative drift, say, for the Nasdaq stock index, whose volatility was \( \sigma = 0.27 \) in the past 10 years.

From (2) we conclude: let \( R - r - \sigma^2 / 2 \) be negative and let \( 0 < b \leq 1 \). Then \( P(\text{Rel}(t) \leq b) \) is a strictly increasing function. In other words, the more we wait, the more likely it is that we will lose a portion of our money. Moreover, this probability approaches 1 (!), since (2) implies the following:

**Observation:** For any \( 0 < b \),

\[
\lim_{t \to \infty} P(\text{Rel}(t) \leq b) = \begin{cases} 0, & \text{if } R - r - \sigma^2 / 2 > 0 \\ 0.5, & \text{if } R - r - \sigma^2 / 2 = 0 \\ 1, & \text{if } R - r - \sigma^2 / 2 < 0. \end{cases}
\]

One can go even further and show that if \( R - r - \sigma^2 / 2 < 0 \), then with a proper (small) \( 0 < c \) value,

\[
P(\lim_{t \to \infty} [\text{Rel}(t)c^t] \leq 1) = 1.
\]

That is, \( \text{Rel}(t) \) tends to zero exponentially fast almost surely. (Here, "almost surely" means with probability 1.) This implies that for large \( t \) with very high probability, \( \text{Rel}(t) \) will be almost zero. However, with a small probability, \( \text{Rel}(t) \) will be huge, since our expected value \( E(\text{Rel}(t)) = e^{(R-r)t} \) is large. This very much resembles a lottery, where we have a tiny chance of winning a huge amount of money, but we lose the price of the ticket almost certainly. (But for the lottery, the expected gain is negative.)

Now let us consider a hypothetical stock market where market participants invest their money in a lump sum only once, and their time horizon is infinity. Even if the drift \( R - r - \sigma^2 / 2 \) is slightly negative, people will be bankrupt in the distant future (maybe a million years) almost surely. But if the drift is positive, almost surely they will be billionaires (or googolplexionaries) because almost surely \( \text{Rel}(t) \) tends to infinity exponentially fast. In this hypothetical world, the drift should be zero, making the risk premium exactly \( \sigma^2 / 2 = 2\% \)!

Here is a discrete example to demonstrate another way that large volatility can cause lottery-like behavior (if the risk premium is insufficient). Suppose we invest $100 for three years, and each year we can either double our money or lose 90 percent of it, with the same probability. After one year our balance will be either $10 or $200. After two years our balance will be $1 or $20 (if our previous balance was $10), or $20 or $400 (if our previous balance was $200). Finally, after three years, our balance will be one of the following, each with the same probability: $0.1, $2, $2, $40, $40, $40, or $800. In one case, we have a huge $700 gain, and in the rest we have big losses. Although our expected gain is $15.76, the final outcome is lottery-like.

**Closing Thoughts**

The model we used is imperfect for several reasons. For instance, we assumed that \( R, r, \) and \( \sigma \) are constants. Even small changes in these parameters can produce large changes in the probabilities we calculated. Also, stock returns have “fat tails” suggesting they are not log-normally distributed.
Finally, we considered the buy-and-hold strategy only, with a pure 100 percent stock portfolio. A mixed bond-stock portfolio would decrease the risk. This author has no idea if stocks will be rising or falling in the coming decades. He keeps 50 percent of his retirement account in stocks and 50 percent in bonds.

Further Reading

A good reference for the mathematical model we used for stock prices is R. J. Williams, *Introduction to the Mathematics of Finance*. See chapter 4.2, beginning on page 57.

Readers can investigate a similar topic, the St. Petersburg paradox, on Wikipedia. Enjoyable readings for beginning investors are *A Random Walk Down Wall Street* and *The Random Walk Guide To Investing*, both by B. G. Malkiel.

*About the author:* David Benko is an assistant professor of mathematics at the University of South Alabama. He has Erdős number two (with multiplicity three) and will represent the American Mathematical Society on the MAA Committee on the American Mathematics Competitions beginning in 2012.

email: dbenko@jaguar1.usouthal.edu

---

**Mathematics Advanced Study Semesters (MASS)**

Department of Mathematics of the Penn State University runs a yearly semester-long intensive program for undergraduate students from across the USA seriously interested in pursuing career in mathematics. MASS is held during the fall semester of each year. For most of its participants, the program is a spring board to graduate schools in mathematics. The participants are usually juniors and seniors.

The MASS program consists of three core courses (4 credits each), Seminar (3 credits) and Colloquium (1 credit), fully transferable to the participants' home schools. The core courses offered in 2010 are:

*Spaces: from geometry to analysis and back* (A. Katok),
*From Euclid to Alexandrov; a guided tour* (A. Petrunin),
*Introduction to Ramsey Theory* (J. Reimann).

Applications for fall semester of 2011 are accepted now.

**Financial arrangements:**
Successful applicants are awarded Penn State MASS Fellowship which reduces their tuition to the in-state level. Applicants who are US citizens or permanent residents receive NSF MASS Fellowship which covers room and board, travel to and from Penn State and provides additional stipend. Applicants with outstanding previous record are awarded additional MASS Merit Fellowship. Participants who significantly exceed expectations during the program will be awarded MASS Performance Fellowships at the end of the semester.

For complete information, see http://www/math.psu.edu/mass
email to mass@math.psu.edu or call (814)865-8402

---

**Further Reading**


*About the authors:* John W. Emerson (Jay) is an associate professor of statistics at Yale University. He has conducted a number of analyses of real-world problems, which typically find their way into his classrooms, and has worked on several studies receiving attention in the popular media.

Silas Meredith is a math and statistics teacher at the Horace Mann School in New York. He is earning a master’s degree in statistics at Columbia University.

email: jayemerson@gmail.com
silas.meredith@gmail.com

The authors would like to thank Miki Seltzer and David Lin for their contributions to earlier work done on this problem, published in the American Statistician in 2009.

DOI: 10.4169/194762111X12954578043019