Logarithms of Integers are Irrational

Jörg Feldvoss*
Department of Mathematics and Statistics
University of South Alabama
Mobile, AL 36688–0002, USA

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Abstract

In this short note we prove that the natural logarithm of every integer ≥ 2 is an irrational number and that the decimal logarithm of any integer is irrational unless it is a power of 10.

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In this short note we prove that logarithms of most integers are irrational.

Theorem 1: *The natural logarithm of every integer $n \geq 2$ is an irrational number.*

Proof: Suppose that $\ln n = \frac{a}{b}$ is a rational number for some integers a and b. Wlog we can assume that $a, b > 0$. Using the third logarithmic identity we obtain that the above equation is equivalent to $n^b = e^a$. Since a and b both are *positive* integers, it follows that e is the root of a polynomial with integer coefficients (and leading coefficient 1), namely, the polynomial $x^a - n^b$. Hence e is an algebraic number (even an algebraic integer) which is a contradiction. $\Box$

Remark 1: It follows from the proof that for any base b which is a *transcendental* number the logarithm $\log_b n$ of every integer $n \geq 2$ is an irrational number. (It is *not* always true that $\log_b n$ is already irrational for every integer $n \geq 2$ if b is *irrational* as the example $b = \sqrt{2}$ and $n = 2$ shows!)

Question: Is it always true that $\log_b n$ is already *transcendental* for every integer $n \geq 2$ if b is *transcendental*?

The following well-known result will be needed in [1, p. 111] and can be proved by a slight variation of the proof of Theorem 1.

Theorem 2: *The decimal logarithm of every integer n is an irrational number unless n is a power of* 10*.*

Proof: Suppose that $\log n = \frac{a}{b}$ is a rational number for some integers a and b. Wlog we can assume that $a, b > 0$. Using the third law for logarithms we obtain that the above equation is equivalent

*E-mail address: `jfeldvoss@jaguar1.usouthal.edu`

to $n^b = 10^a = 2^a \cdot 5^a$. Since a and b both are *positive* integers, it follows from the Fundamental Theorem of Arithmetic that $n = 2^r \cdot 5^s$ for some positive integers r and s. Furthermore, by the same token $2^{rb} \cdot 5^{sb} = 2^a \cdot 5^a$ implies by that $rb = a$ and $sb = a$, i.e., $r = s$. Hence $n = 2^r \cdot 5^r = 10^r$. $\square$

Remark 2: It follows along the same lines as in the proof of Theorem 2 that for any integer base b which is a product of distinct prime numbers the logarithm $\log_b n$ of every integer n is an irrational number unless n is a power of b. (This is *not* true if b contains some prime factor at least twice as the example $\log_4 8 = \frac{3}{2}$ shows. I am grateful to Joshua Haber for pointing this out to me.)

References

[1] Boris Hasselblatt and Anatole Katok: *A First Course in Dynamics with a Panorama of Recent Developments*, Cambridge University Press, Cambridge/New York, 2003.