Let \( g \) be a Lie algebra and let \( E \) be a vector space that contains \( g \) as a subspace. The paper under review deals with a classification of all Lie algebra structures \([ -,-]\) on \( E \) such that \( g \) is a Lie subalgebra of \((E, [ -,-])\) (for short, \((E, [ -,-])\) extends \( g \)) up to an isomorphism of Lie algebras that is compatible with the inclusion and projection maps involved. This generalizes the classical extension problem, where \( g \) is an ideal of \( E \) and the factorization problem which leads to a bicrossproduct construction.

The existence part is solved by the construction of a unified product \( g \gtimes V \) for some vector space complement \( V \) of \( g \) in \( E \). More generally, let \( V \) be any vector space over the same ground field as \( g \) and consider an extending datum \( \Omega(g, V) = (\langle \cdot, \cdot \rangle, \triangleright, f, \{ -,-\}) \) of \( g \) through \( V \) consisting of bilinear maps \( \langle \cdot \rangle : V \times g \to V \), \( \triangleright : V \times V \to g \), and \( \{ -,-\} : V \times V \to V \). For any extending datum \( \Omega(g, V) \) the authors define a bilinear map on \( g \times V \) and prove that this map turns \( g \times V \) into a Lie algebra if, and only if, the ingredients of \( \Omega(g, V) \) satisfy certain compatibility conditions (in particular, \( V \) is a right \( g \)-module via \( \langle \cdot \rangle \) and a twisted left \( g \)-module via \( \triangleright \), \( f \) is a twisted (alternating) cocycle for \( V \) and \( \{ -,-\} \) is a twisted (alternating) Lie bracket on \( V \)). In this case they call \( \Omega(g, V) \) a Lie extending structure of \( g \) through \( V \) and \( g \gtimes V \) a unified product (unifying the twisted semi-product occurring in the extension problem and the bicrossproduct arising in the factorization problem) of \( g \) by \( V \). Now all Lie algebra structures \((E, [ -,-])\) extending \( g \) can be described as follows. If \( V \) is vector space complement of \( g \) in \( E \), and if \( [ -,-] \) is a Lie algebra structure on \( E \) such that \( g \) is a Lie subalgebra of \((E, [ -,-])\), then there exists a Lie extending structure \( \Omega(g, V) = (\langle \cdot, \cdot \rangle, \triangleright, f, \{ -,-\}) \) of \( g \) through \( V \) such that \((E, [ -,-])\) is isomorphic to \( g \gtimes V \) as Lie algebras and the isomorphism is compatible with both the inclusion and projection maps involved.

The classification part is given by a pointed quotient set \( H_g(V, g) \) of the set of all Lie extending structures \( \mathcal{L}(g, V) \) of \( g \) through a vector space complement \( V \) of \( g \) in \( E \) by a suitable equivalence relation. Then there is a bijection between the set \( H_g(V, g) \) and the set of equivalence classes of all Lie algebra structures \((E, [ -,-])\) extending \( g \), where two extending Lie algebra structures are equivalent if they are isomorphic and the isomorphism is compatible with the inclusion maps involved. Moreover, the set
$\mathcal{H}_g(V, g)$ is a quotient of a cohomological object $\mathcal{H}(V, g)$ which generalizes the second Chevalley-Eilenberg cohomology group for Lie algebras and is in bijection with the set of equivalence classes of all Lie algebra structures $(E, [-, -])$ extending $g$, where two extending Lie algebra structures are equivalent if they are isomorphic and the isomorphism is compatible with both the inclusion and projection maps involved.

The paper closes with some examples and applications of these constructions. In particular, for so-called flag-extending structures (i.e., if there is a flag $g = E_0 \subset E_1 \subset \cdots \subset E_m = E$ of Lie subalgebras $E_0$, $E_1$, ..., $E_m$ of codimension 1 in each other, the authors show that $\mathcal{H}_g(V, g)$ and $\mathcal{H}(V, g)$ can be parametrized by twisted derivations, and then in two examples (in one example $g$ is perfect and in the other one $g$ is not perfect) they explicitly describe all Lie algebra structures $(E, [-, -])$ extending $g$ for which the codimension of $g$ in $E$ is 1.