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k-modules over linear spaces by \(n\)-linear maps admitting a multiplicative
basis.

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Let \(n\) be a positive integer, let \(S_n\) denote the symmetric group on the set \(\{1, \ldots, n\}\), and fix an integer \(k\) with \(1 \leq k \leq n\). For an arbitrary field \(F\) the authors say that an \(F\)-vector space \(V\) is a \(k\)-module over an \(F\)-vector space \(W\) if for each of the \(\binom{n}{k}\) distinct permutations \(\sigma\) of \(V^k \times W^{n-k}\) there is an \(n\)-multilinear map from the corresponding Cartesian product \((V^k \times W^{n-k})_\sigma\) to \(V\). One example for a 1-module is a two-sided module \(M\) over an \(F\)-algebra \(A\), i.e., an \(F\)-vector space \(M\) with a right \(F\)-bilinear \(A\)-action \(\rho: M \times A \to M\) and a left \(F\)-bilinear \(A\)-action \(\lambda: A \times M \to M\). (Note that in representation theory one also requires compatibility conditions between the right and the left action.) Another example is an \(n\)-algebra over \(F\) that is an \(F\)-vector space \(A\) with an \(n\)-multilinear map \(A^n \to A\). (The usual algebras are 2-algebras and triple systems are 3-algebras, etc.)

The main goal of the paper under review is to study \(k\)-modules admitting a multiplicative basis, where the latter is defined as follows. Let \(V\) be a \(k\)-module over \(W\), and let \(B' = \{w_j \mid j \in J\}\) be a basis of the \(F\)-vector space \(W\). Then a basis \(B = \{v_i \mid i \in I\}\) of the \(F\)-vector space \(V\) is called multiplicative with respect to \(B'\) if for any \(\sigma \in S_n\), any \(i_1, \ldots, i_k \in I\), and any \(j_{k+1}, \ldots, j_n \in J\) the image \([v_{i_1}, \ldots, v_{i_k}, w_{j_{k+1}}, \ldots, w_{j_n}]_\sigma\) of \((v_{i_1}, \ldots, v_{i_k}, w_{j_{k+1}}, \ldots, w_{j_n})_\sigma \in (V^k \times W^{n-k})_\sigma\) under the \(n\)-multilinear map corresponding to \(\sigma\) is a scalar multiple of \(v_{i'}\), for some unique \(r_\sigma \in I\). This generalizes the concept of a multiplicative basis of an \(n\)-algebra introduced by A. J. Calderón Martín, F. J. Navarro Izquierdo, and J. M. Sánchez Delgado [J. Algebra Appl. 17 (2018), no. 2, 1850025, 11 pp.; MR3749483] and several other related concepts. In the first example mentioned above, let \(B' = \{a_j \mid j \in J\}\) be a basis of the \(F\)-algebra \(A\). Then a basis \(B = \{m_i \mid i \in I\}\) of the 1-module \(M\) is multiplicative with respect to \(B'\) if for every \(i \in I\) and every \(j \in J\) we have that \(\rho(m_i, a_j)\) is a scalar multiple of \(m_{i'}\) for a unique \(i' \in I\) and \(\lambda(a_j, m_i)\) is a scalar multiple of \(m_{i'}\) for a unique \(m_{i'} \in I\).

The main results in this paper are: 1) if a \(k\)-module admits a multiplicative basis, then it is a direct sum of \(k\)-submodules admitting a multiplicative basis, 2) under mild conditions the direct summands in 1) are in a certain sense minimal, and 3) an application of \(k\)-modules admitting a multiplicative basis \(B\) with respect to \(B'\) over an \(n\)-algebra admitting the multiplicative basis \(B'\).