Let $\mathfrak{g}$ be a root-reductive Lie algebra over an algebraically closed field of characteristic zero. (Recall that a root-reductive Lie algebra is a direct limit of finite-dimensional reductive Lie algebras which preserves their root spaces, for example, the locally finite-dimensional Lie algebra $\mathfrak{gl}(\infty)$ of infinite matrices with only finitely many non-zero entries.) The goal of the paper under review is to introduce and study a certain analogue of the BGG category $\mathcal{O}$ for root-reductive Lie algebras. The definition of a BGG category depends on the choice of a Borel subalgebra. Contrary to the classical situation, where two Borel subalgebras are conjugate, in general, a root-reductive Lie algebra $\mathfrak{g}$ has uncountably many conjugacy classes of Borel subalgebras. Known results for integrable $\mathfrak{g}$-modules and primitive ideals in $U(\mathfrak{g})$ suggest to single out two mutually exclusive classes of Borel subalgebras, namely, the Dynkin Borel subalgebras and the ideal Borel subalgebras. The latter were used by the second author and Petukhov [On ideals in $U(\mathfrak{sl}(\infty)), U(\mathfrak{o}(\infty)), U(\mathfrak{sp}(\infty))$, in: Representation Theory – Current Trends and Perspectives, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2017, 565–602; MR3644805] and [Bull. Lond. Math. Soc. 50 (2018), no. 3, 435–448; MR3829731] to classify the primitive ideals of $U(\mathfrak{sl}(\infty))$. In his Ph.D. thesis and in the paper [On categories $\mathcal{O}$ for root-reductive Lie algebras, preprint arXiv:1711.11234v1 [math.RT]] Nampaisarn defined and studied a BGG category $\mathcal{O}$ for a root-reductive Lie algebra $\mathfrak{g}$ using a Dynkin Borel subalgebra $\mathfrak{b}$, where the latter is a maximal locally solvable subalgebra of $\mathfrak{g}$ that contains a certain “nice” maximal toral subalgebra $\mathfrak{h}$ and is generated by $\mathfrak{h}$ and the simple root spaces of $\mathfrak{g}$. Note that a Borel subalgebra of $\mathfrak{g}$ that contains $\mathfrak{h}$ is a Dynkin Borel subalgebra if, and only if, every Verma module has only finite-dimensional weight spaces. Moreover, the category $\mathcal{O}$ is not a highest weight category as it neither has enough projective nor enough injective modules.

In the paper under review the authors define two slightly different BGG categories $\mathcal{O}$ and $\mathcal{O}$ such that $\mathcal{O} \subseteq \mathcal{O} \subseteq \mathcal{O}$. Here $\mathcal{O}$ is the category of all $\mathfrak{g}$-modules that are $\mathfrak{h}$-semisimple and on which $\mathfrak{b}$ acts locally finitely, and $\mathcal{O}$ is the full subcategory of $\mathcal{O}$ that only contains finitely generated $\mathfrak{g}$-modules. In particular, Verma modules are objects of all three categories. The authors prove many useful facts about these categories: 1) $\mathcal{O}$ is a Grothendieck category (i.e., an abelian category with arbitrary
coproducts and a generator such that direct limits of short exact sequences are exact), and, as a consequence, has enough injective modules. 2) The simple objects in $\mathcal{O}$ are (up to isomorphism) exactly the simple highest weight modules. 3) $\mathcal{O}$ decomposes into blocks parametrized by orbits of the Weyl group in the linear dual $\mathfrak{h}^*$ of $\mathfrak{h}$. 4) $\mathcal{O}$ contains certain Serre subcategories $^K\mathcal{O}$ (i.e., non-empty full subcategories $\mathcal{S}$ such that the middle term of a short exact sequence is an object of $\mathcal{S}$ if, and only if, the other two terms are objects of $\mathcal{S}$) that are obtained by truncation to an upper finite ideal $\mathcal{K}$ of $\mathfrak{h}^*$ and which have enough projective modules. 5) $^K\mathcal{O}$ is extension full in $\mathcal{O}$ and also in the larger category of all $\mathfrak{h}$-semisimple $\mathfrak{g}$-modules. 6) $^K\mathcal{O}$ has a graded cover in which there are graded lifts of simple modules, Verma modules, and dual Verma modules. These satisfy a property that in the case of the classical category $\mathcal{O}$ implies Koszulity. 7) $\mathcal{O}$ is Ringel self-dual.