The knowledge of the representations of least dimension $> 1$ for a given Lie algebra has several applications. In particular, it is useful in determining the maximal subalgebras of a Lie algebra or the maximal subgroups of a Lie group. In [Convolution and equidistribution: Sato-Tate theorems for finite-field Mellin transforms, Annals of Mathematics Studies, vol. 180, Princeton University Press, Princeton, NJ, 2012. viii+203 pp.; MR2850079] N. M. Katz asked to find those finite-dimensional complex simple Lie algebras that have non-self-dual irreducible representations of dimension $pq$, where $p$ and $q$ are two distinct prime numbers.

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra of rank $n$. Then the finite-dimensional irreducible $\mathfrak{g}$-modules are in bijection with the dominant weights of a Cartan subalgebra of $\mathfrak{g}$, or equivalently, of a maximal torus of $\mathfrak{g}$. Every dominant weight can be uniquely written as a linear combination of the $n$ fundamental weights with non-negative integers as coefficients. The sum of these coefficients is called the width of the dominant weight. The first main result of the paper under review determines, among all dominant weights $\omega$ of a given width, the smallest dimension of an irreducible $\mathfrak{g}$-module $V(\omega)$ of highest weight $\omega$, and the dominant weight for which this minimum is achieved.

Let $(-, -)$ be an appropriately scaled bilinear form induced from the Killing form of $\mathfrak{g}$ on $\mathfrak{g}^* \times \mathfrak{g}^*$. It follows from Weyl’s dimension formula that the largest prime number dividing $\dim V(\omega)$ is at most $(\omega + \rho, \alpha)$, where $\rho$ denotes the sum of the fundamental weights, and $\alpha$ is either the highest positive root of $\mathfrak{g}$ for Lie algebras of types different from $B$ or 2 times the highest short root for Lie algebras of type $B$. In particular, if $\dim V(\omega)$ is the product of at most two prime numbers, then one has that $\dim V(\omega) \leq (\omega + \rho, \alpha)^2$. The second main result of the paper under review classifies those Lie algebras $\mathfrak{g}$ of rank at least 2 and those dominant weights $\omega$ for which $\dim V(\omega) \leq (\omega + \rho, \alpha)^2$. As a consequence, the authors answer the question of Katz. In addition, they also give another proof for Gabber’s classification of all irreducible $\mathfrak{g}$-modules of prime dimension. (Reviewer’s remarks: It should be noted that in stating Gabber’s result there is a typo in the paper, namely, $n - 1$ should be replaced by $n + 1$. Moreover, the authors forgot to mention that for the Lie algebra of type $A_1$ the $(p - 1)^{th}$ symmetric power of the two-dimensional natural module has prime dimension $p$ for any prime number $p$, and for $p \neq 2$ this module is not isomorphic to the natural module itself.)