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In the paper under review the authors investigate the homological finiteness conditions $FP_m$ and $FP_\infty$ for Lie algebras. Let $L$ be a Lie algebra over a field $K$ of arbitrary characteristic and let $U(L)$ denote the universal enveloping algebra of $L$. Then every $L$-module is a unital $U(L)$-module and especially the one-dimensional trivial $L$-module $K$ is a unital $U(L)$-module. An $L$-module $M$ is said to be of type $FP_m$ if $M$ has a projective resolution $\cdots \to P_i \to \cdots \to P_1 \to P_0 \to M \to 0$ as a unital $U(L)$-module such that $P_i$ is finitely generated for every $i \leq m$ and $M$ is of type $FP_\infty$ if $M$ has a projective resolution such that all terms are finitely generated. Finally, $L$ is of type $FP_m$ (resp. of type $FP_\infty$) if the trivial $L$-module $K$ is of type $FP_m$ (resp. of type $FP_\infty$).

The authors show that solvable Lie algebras of type $FP_\infty$ are finite-dimensional by using similar methods as P. H. Kropholler [Bull. London Math. Soc. 25 (1993), no. 6, 558–566; MR1245082 (94j:20051a)] who proved that solvable groups of type $FP_\infty$ over the rational numbers have finite torsion-free rank. Moreover, they obtain that abelian-by-finite-dimensional Lie algebras of type $FP_m$ are finite-dimensional if either $0 < \text{char}(K) \leq m$ or if the dimension of the finite-dimensional factor is strictly less than $m$. The first condition generalizes a result obtained by the first author and R. M. Bryant [J. Algebra 218 (1999), no. 1, 1–25; MR1704674 (2001a:17013)].

In the last section of the paper the authors define a certain class $H(\{0\})$ of Lie algebras similar to the classes of groups considered by P. H. Kropholler [J. Pure Appl. Algebra 90 (1993), no. 1, 55–67; MR1246274 (94j:20051b)]. The definition of the classes of groups which Kropholler considers uses actions of groups on cell complexes. In the paper under review this is replaced by an appropriate description of a projective resolution of the ground field of the Lie algebra. Let $\mathcal{X}$ be a class of Lie algebras over the field $K$. Set $H_0(\mathcal{X}) := \mathcal{X}$ and for any ordinal $\alpha > 0$ define $H_\alpha(\mathcal{X})$ to be the class consisting of those Lie algebras $L$ over $K$ for which there exists an exact sequence $0 \to P_s \to \cdots \to P_1 \to P_0 \to K \to 0$ such that $P_i$ is a (possibly infinite) direct sum of induced modules $U(L) \otimes_{U(H)} K$ (for (possibly different) subalgebras $H$ of $L$ that are in $\bigcup_{\beta < \alpha} H_\beta(\mathcal{X})$). Finally, set $H(\mathcal{X}) := \bigcup_\alpha H_\alpha(\mathcal{X})$. The authors show that the class $H(\{0\})$ is closed under extensions and countable directed unions. In particular,
$H(\{0\})$ contains all solvable Lie algebras of countable dimension. But it does not seem to be known whether, as in the corresponding case for groups, $H(\{0\})$ is closed under taking subalgebras. Nevertheless, the authors are able to show that modules of type $FP_\infty$ over Lie algebras in $H(\{0\})$ have finite projective dimension. As in the group case the proof uses the complete cohomology introduced by P. Vogel (cf. F. Goichot [J. Pure Appl. Algebra 82 (1992), no. 1, 39–64; MR1181092 (94d:55014)]) and axiomatized by G. Mislin [Topology Appl. 56 (1994), no. 3, 293–300; MR1269317 (95c:20072)].