Let $K$ be an algebraically closed field of prime characteristic $p$, and let $n$ be a positive integer. The paper discusses the restriction of simple modules of the general linear Lie algebra $\mathfrak{gl}_n(K)$ of all $n \times n$ matrices with coefficients in $K$ to the subalgebra $\mathfrak{sl}_n(K)$ of all $n \times n$ matrices with trace zero. The first main result states that this restriction either is itself simple or it has precisely $p$ composition factors that are all isomorphic to each other.

In order to be able to formulate the second main result of the paper, we first need to introduce some more notation. Since $\mathfrak{gl}_n(K)$ is a restricted Lie algebra via the usual $p$-th power of a matrix and the ground field $K$ is algebraically closed, every simple $\mathfrak{gl}_n(K)$-module $E$ has a $p$-character $\chi$, i.e., there exists a linear form $\chi$ on $\mathfrak{gl}_n(K)$ such that for every element $X$ of $\mathfrak{gl}_n(K)$ the element $X^p - X^{[p]} - \chi(X)p$ of the universal enveloping algebra $U(\mathfrak{gl}_n(K))$ of $\mathfrak{gl}_n(K)$ annihilates $E$, where $X^p$ denotes the power of $X$ in $U(\mathfrak{gl}_n(K))$ and $X^{[p]}$ denotes the power of $X$ as a matrix. Moreover, as the bilinear form $(X, Y) \mapsto \text{tr}(XY)$ on $\mathfrak{gl}_n(K)$ is non-degenerate, for every linear form $\chi$ on $\mathfrak{gl}_n(K)$ there exists a unique matrix $C_\chi$ in $\mathfrak{gl}_n(K)$ such that $\chi(X) = \text{tr}(C_\chi X)$ for any matrix $X$ in $\mathfrak{gl}_n(K)$. The second main result of the paper characterises in terms of the matrix $C_\chi$ when each case of the first main result occurs for simple $\mathfrak{gl}_n(K)$-modules with a fixed $p$-character $\chi$. Namely, if all blocks in the Jordan normal form of $C_\chi$ have size divisible by $p$, then up to isomorphism there is a unique simple $\mathfrak{gl}_n(K)$-module with $p$-character $\chi$ such that the restriction to $\mathfrak{sl}_n(K)$ is not simple. Otherwise, the restriction of every simple $\mathfrak{gl}_n(K)$-module with $p$-character $\chi$ to $\mathfrak{sl}_n(K)$ is simple. In particular, the restrictions of simple restricted $\mathfrak{sl}_n(K)$-modules to $\mathfrak{sl}_n(K)$ are always simple. The last result was known before since simple restricted $\mathfrak{gl}_n(K)$-modules as well as simple restricted $\mathfrak{sl}_n(K)$-modules come from simple modules of the corresponding algebraic groups $\text{GL}_n(K)$ and $\text{SL}_n(K)$, respectively, and the restrictions of simple $\text{GL}_n(K)$-modules to $\text{SL}_n(K)$ are always simple. Moreover, in the other extreme case that $\chi$ is regular nilpotent (or equivalently, $C_\chi$ has one Jordan block of size $n$) and $p$ divides $n$, every baby Verma module $\mathcal{Z}_\chi(\lambda)$ for $\mathfrak{gl}_n(K)$ is simple, and a result proved by author, from which the second main result follows, shows that the restriction of $\mathcal{Z}_\chi(0)$ to $\mathfrak{sl}_n(K)$ is not simple. This was already observed by Xin Wen [J. Algebra Appl., to appear] in the special case $p = n = 3$ and motivated the paper under review.