Let $R$ be a commutative ring with unity element $1_R$, and let $\mathfrak{gl}_n(R)$ denote the general linear Lie algebra over $R$ with natural basis $\{e_{ij} \mid 1 \leq i, j \leq n\}$. For any partial ordering $\preceq$ on $\{1, \ldots, n\}$ let $\mathfrak{gl}_{\preceq}^n(R)$ denote the Lie subalgebra of $\mathfrak{gl}_n(R)$ with basis $\{e_{ij} \mid i \preceq j\}$. Note that, if $\preceq$ is the usual total order $\leq$ on $\{1, \ldots, n\}$, then $\mathfrak{gl}_{\preceq}^n(R)$ is the solvable Lie algebra $\mathfrak{so}_n(R)$ of upper triangular matrices over $R$.

In the paper under review algebraic discrete Morse theory developed by Michael Jöllenbeck in his Ph.D. thesis and independently by Emil Sköldberg in [Trans. Amer. Math. Soc. 358 (2006), no. 1, 115–129; MR2171225] is used to compute the (co)homology of the Lie algebras $\mathfrak{gl}_{\preceq}^n(R)$ with trivial coefficients. More precisely, if either every integer multiple of $1_R$ is invertible or if the characteristic $p$ of $R$ is sufficiently large (precisely, if $p \geq n$), then the cohomology of $\mathfrak{gl}_{\preceq}^n(R)$ with coefficients in $R$ as a graded $R$-algebra is isomorphic to the exterior algebra over $R$ in $n$ generators of degree 1. In these cases, or if the characteristic $p$ of $R$ is large compared to the degree $k$ of the (co)homology space (precisely, if $p > \frac{k+1}{2}$), then $H_k(\mathfrak{gl}_{\preceq}^n(R), R) \cong H^k(\mathfrak{gl}_{\preceq}^n(R), R)$ is isomorphic to $R(\binom{n}{k})$ as an $R$-module. In particular, under these assumptions the (co)homology of $\mathfrak{gl}_{\preceq}^n(R)$ with coefficients in $R$ is a free $R$-module in every degree $k$.

In degree 1 and 2 this also holds for $R = \mathbb{Z}$, but already $H_3(\mathfrak{gl}_{\preceq}^n(\mathbb{Z}), \mathbb{Z})$ has torsion. Moreover, in prime characteristic the authors compute $H_k(\mathfrak{so}_n(R), R)$ in the first cases $p = n - 1$ or $p = \frac{k+1}{2}$, for which the above result does not apply, and they determine when $H_k(\mathfrak{so}_n(\mathbb{Z}_p), \mathbb{Z}_p)$ vanishes in high degrees.

There are also several other results. One is that the (co)homology of $\mathfrak{gl}_{\preceq}^{n-1}(R)$ is a direct summand (factor) of the (co)homology of $\mathfrak{gl}_{\preceq}^n(R)$. Another is that for every interval $\{x_0, x_1, \ldots, x_t\}$ in the poset $\{1, \ldots, n\}$ the $\mathbb{Z}$-module $H_{2t-3}(\mathfrak{gl}_{\preceq}^n(\mathbb{Z}), \mathbb{Z})$ has a direct summand isomorphic to $\mathbb{Z}_t$. Furthermore, the authors also relate the homology $H_k(\mathfrak{gl}_{\preceq}^n(\mathbb{Z}_p), \mathbb{Z}_p)$ to the homology of the Lie algebra $\mathfrak{gl}_{\preceq}^n(\mathbb{Z}_p)$ of poset strictly triangular matrices with basis $\{e_{ij} \mid i < j\}$. 