Ooms, Alfons I.: 
The Gelfand-Kirillov conjecture for semi-direct products of Lie algebras. 

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Let \( g \) be a finite-dimensional Lie algebra over a field of characteristic zero and let \( W \) be a \( g \)-module with \( \dim W \geq \dim g \). The main result of the paper under review gives sufficient conditions for the semi-direct product \( g \oplus W \) of \( g \) and \( W \) to satisfy the Gelfand-Kirillov conjecture (GK). In particular, this implies that GK holds for a certain class of Frobenius Lie algebras.

The author also discusses semi-direct products of \( \mathfrak{sl}_2 \) with direct sums of irreducible \( \mathfrak{sl}_2 \)-modules. In the following let \( W_n \) denote the \((n+1)\)-dimensional irreducible \( \mathfrak{sl}_2 \)-module. The author proved in a previous paper with J. Alev and M. Van den Bergh [J. Algebra **227** (2000), no. 2, 549–581; MR1759835 (2001e:17017a)] that GK is true for all algebraic Lie algebras of dimension at most eight over an algebraically closed field of characteristic zero. This is the best one can hope for since it follows from the main result of another paper by the author, J. Alev, and M. Van den Bergh [Trans. Amer. Math. Soc. **348** (1996), no. 5, 1709–1716; MR1321564 (96h:17011)] that the semi-direct product \( \mathfrak{sl}_2 \oplus (W_1 \oplus W_2) \) of \( \mathfrak{sl}_2 \) with two copies of its adjoint module is a nine-dimensional counterexample to GK. In the paper under review the method of the proof of its main result is illustrated by finding explicit generators for the quotient division ring of the universal enveloping algebra of the semi-direct product \( \mathfrak{sl}_2 \oplus (W_1 \oplus W_2) \). Furthermore, in the case of an algebraically closed ground field the author proves that \( \mathfrak{sl}_2 \oplus W_n \) satisfies GK for \( n \geq 5 \) if and only if \( n \) is odd. In particular, the case \( n = 6 \) gives a ten-dimensional counterexample to GK. This also settles completely the validity of GK for semi-direct products of \( \mathfrak{sl}_2 \) with one irreducible \( \mathfrak{sl}_2 \)-module since it was known before by the two papers mentioned above that for \( n \leq 4 \) the semi-direct product \( \mathfrak{sl}_2 \oplus W_n \) always satisfies GK.