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Maximal subalgebras of Cartan type in the exceptional Lie algebras.

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In the early 1950's Dynkin classified the maximal subalgebras of finite-dimensional simple Lie algebras over the complex numbers. The paper under review initiates the study of maximal subalgebras of exceptional simple classical Lie algebras in good characteristics. More precisely, let $G$ be a simple algebraic group of exceptional type over an algebraically closed field of characteristic $p$ such that $p$ is a good prime for the root system of $G$, and let $\mathfrak{g}$ denote the Lie algebra of $G$. Note that every maximal subalgebra of $\mathfrak{g}$ is a $p$-subalgebra (i.e., it is closed under the $p$-map of the restricted Lie algebra $\mathfrak{g}$) as $\mathfrak{g}$ has no non-zero proper ideals.

The authors answer two related questions on the subalgebra structure of the Lie algebra $\mathfrak{g}$. Firstly, they prove the existence of a $p$-subalgebra isomorphic to the $p$-dimensional Witt algebra $W_1$ and characterise when this $p$-subalgebra is maximal. This essentially is done by classifying the nilpotent elements $e$ in $\mathfrak{g}$ that correspond to the element of degree $-1$ in the standard basis of $W_1$. In particular, if the $p$-subalgebra isomorphic to $W_1$ is maximal, then $e$ is regular, and conversely, if $e$ is regular and $\mathfrak{g}$ is not of type $E_6$, then the $p$-subalgebra isomorphic to $W_1$ is maximal. Secondly, it is proved that every simple subalgebra of $\mathfrak{g}$ is either isomorphic to $W_1$ or it is of classical type. Roughly, the reason that non-classical simple Lie algebras cannot fit into exceptional simple classical Lie algebras in good characteristics is the large dimensions of the former compared to the dimensions of the latter.

The proof of the first result needs several explicit computations which are done using GAP and are described in an appendix. The proof of the second result uses the Block-Wilson-Premet-Strade classification of finite-dimensional simple Lie algebras over an algebraically closed field of characteristic $p > 3$. Moreover, some techniques are needed to classify certain irreducible representations of non-graded Hamiltonian Lie algebras of absolute toral rank two that only recently were developed in a paper by the reviewer, Salvatore Siciliano, and Thomas Weigel [Transform. Groups 21, No. 2, 377-398 (2016; Zbl. 06592114)].