Premet, Alexander; Strade, Helmut
Simple Lie algebras of small characteristic. VI: Completion of the classification.

Mathematics Subject Classification 2000: *17B50; 17B20

Keywords: Classification of simple Lie algebras; small characteristic; \( p \)-envelope; torus of maximal dimension; root; standard torus; Lie algebra of classical type; Lie algebra of Cartan type; generalized Kostrikin-Shafarevich conjecture; non-standard torus; triangulability; Melikian algebra; non-degenerate invariant bilinear form; central extension; irreducible module

Reviewer: Jörg Feldvoss (8086)

This paper is the final one in a series of papers on the classification of finite-dimensional simple Lie algebras over an algebraically closed field of prime characteristic \( p > 3 \). Let \( L \) be a finite-dimensional simple Lie algebra over an algebraically closed field of prime characteristic \( p > 3 \) and let \( T \) be a torus of maximal dimension in the minimal \( p \)-envelope \( L_p \) of \( L \) in its derivation algebra. In the fourth part [J. Algebra 278, No. 2, 766-833 (2004; Zbl. 1155.17306)] the authors showed that only four types of roots of \( L \) relative to \( T \) can occur, namely solvable, classical, Witt, and Hamiltonian roots (meaning that the semisimple quotient of the corresponding 1-section is either 0 or contains a copy of \( \mathfrak{sl}(2) \) or the Witt algebra or the smallest restricted simple Hamiltonian Lie algebra as an ideal of codimension at most 1). Moreover, they show that \( L \) is either of classical type or a Block algebra or a filtered Lie algebra of Cartan type \( S \) if all roots are either solvable or classical. In the fifth part [J. Algebra 314, No. 2, 664-692 (2007; Zbl. 1139.17007)] the authors prove that \( L \) is a filtered Lie algebra of Cartan type if any torus of maximal dimension in \( L_p \) is standard (i.e., the derived subalgebra of its centralizer in \( L \) acts nilpotently on \( L \)) and the set of roots of such a torus in \( L_p \) contains either a Witt root or a Hamiltonian root. This in conjunction with the results from the fourth part yields that if \( L \) is a finite-dimensional simple Lie algebra over an algebraically closed field of prime characteristic \( p > 3 \) such that any torus of maximal dimension in \( L_p \) is standard, then \( L \) is either of classical or Cartan type. In particular, the generalized Kostrikin-Shafarevich conjecture holds for \( p = 7 \). One main ingredient in the proof of this result and the main theorem of the paper under review is Premet’s generalization of Wilson’s theorem on the triangulability of
the Cartan subalgebras of finite-dimensional simple Lie algebras [J. Algebra 167, No. 3, 641-703 (1994; Zbl. 0802.17012)] which in particular implies that non-standard tori in $L_p$ only occur if $p = 5$ and $L$ is isomorphic to a Melikian algebra.

In the present paper the authors consider the remaining case where $p = 5$ and $L_p$ contains a non-standard torus of maximal dimension. The main result is that then $L$ is isomorphic to a Melikian algebra. This result completes the classification of the finite-dimensional simple Lie algebras over an algebraically closed field of prime characteristic $p > 3$.

Let me conclude by describing in more detail some of the ingredients of the proof of the main result of the paper under review. The crucial result of the present paper is a refinement of the local analysis of the 2-sections with core $H(2; (2, 1))^{(2)}$ from the previous part for $p = 5$, which relies heavily on the knowledge of the conjugacy classes of certain toral derivations of the non-restricted Hamiltonian Lie algebra $H(2; (2, 1))$. Moreover, some information on the structure and the representations of Melikian algebras is needed that was not known previously. Namely, the authors prove that every Melikian algebra admits a non-degenerate symmetric invariant bilinear form and that every central extension of the 125-dimensional restricted Melikian algebra $M(1, 1)$ is trivial. The authors also show that the smallest non-restricted irreducible $M(1, 1)$-module has dimension 125. This is of interest in its own and gives important insight into the $p$-mapping of $L_p$. The last ingredient is the existence of a non-standard torus of maximal dimension in $L_p$ for which the third term of the descending central series of its centralizer does not contain non-zero toral elements.