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The tensor product of representations of $U_q(\mathfrak{sl}_2)$ via quivers.

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The main goal of the present paper is to obtain a geometric realization of the tensor product of a finite number of finite-dimensional irreducible representations of $U_q(\mathfrak{sl}_2)$ using quiver varieties. A quiver variety corresponding to the tensor product of finitely many integrable highest weight modules of a Kac-Moody algebra with a simply laced root system was defined by H. Nakajima [Invent. Math. 146, No. 2, 399-449 (2001; Zbl. 1023.17008)] and A. Malkin [Duke Math. J. 116, No. 3, 477-524 (2003; Zbl. 1048.20029)]. The paper under review considers the simplest case $\mathfrak{sl}_2$ and recovers the complete structure of $U_q(\mathfrak{sl}_2)$ via the tensor product variety and not just its crystal structure as in the more general cases.

In the paper under review the tensor product variety $\mathcal{T}(d)$ is defined over the finite field $\mathbb{F}_{q^2}$ with $q^2$ elements rather than over the complex numbers. The author finds three spaces of invariant functions on $\mathcal{T}(d)$ which all are isomorphic to the tensor product $V_{d_1} \otimes \cdots \otimes V_{d_k}$ of finite-dimensional simple $U_q(\mathfrak{sl}_2)$-modules. The natural bases of these spaces correspond to the elementary basis $B_e$, Lusztig’s canonical basis $B_c$, and a basis $B_s$ compatible with the decomposition of $V_{d_1} \otimes \cdots \otimes V_{d_k}$ into a direct sum of simple modules, respectively. The bases $B_e$ and $B_s$ are characterized by their relation to the irreducible components of $\mathcal{T}(d)$. Here an irreducible component of $\mathcal{T}(d)$ defined over $\mathbb{F}_{q^2}$ consists of the $\mathbb{F}_{q^2}$-rational points of the irreducible component of the corresponding variety defined over the algebraic closure of $\mathbb{F}_{q^2}$. Elements of the bases $B_e$ and $B_s$ are equal to a non-zero constant on the set of dense points of an irreducible component of $\mathcal{T}(d)$ with supports contained in distinct irreducible components. However, the elements of $B_s$ have disjoint supports unlike the elements of $B_e$.

The paper contains also a geometric description of the space of intertwiners $\text{Hom}_{U_q(\mathfrak{sl}_2)}(V_{d_1} \otimes \cdots \otimes V_{d_k}, V_\lambda)$ and the natural basis of this space is again characterized by its relation to the irreducible components of $\mathcal{T}(d)$.

An important tool used in this paper is the Penrose-Kauffman graphical calculus of intertwiners for $U_q(\mathfrak{sl}_2)$ expanded by I. B. Frenkel and M. G. Khovanov [Duke Math. J. 87, No. 3, 409-480 (1997; Zbl. 0883.17013) in order to prove various results for Lusztig’s (dual) canonical basis.