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Simple Lie algebras over fields of positive characteristic. III: Completion of the classification.

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Reviewer: Jörg Feldvoss (8086)

The book under review is the last of three volumes devoted to a unified proof of the classification of finite-dimensional simple Lie algebras over an algebraically closed field of characteristic $p > 3$. The goal of the third volume is the classification of simple Lie algebras of absolute toral rank greater than two. Together with the first volume, which in addition to some foundational material contained the classification of the simple Lie algebras of absolute toral rank one, and the second volume, which was dealing with the classification of the simple Lie algebras of absolute toral rank two, the last volume will complete the classification in the sense that a list of simple Lie algebras is given and it is proved that this list is complete.

In more detail the third volume contains the following results. (Note that contrary to the first volume the ground field in the third volume is always assumed to be an algebraically closed field of characteristic $p > 3$.) In Chapter 16 (the first chapter of the third volume) the author discusses several different topics which will be needed later. The first two sections contain material related to tori in Lie algebras of Cartan type. In particular, it is discussed how the orbits of semisimple elements in the Hamiltonian algebra $H(2; 1; \Phi(1))$ fit into a two-dimensional torus of its minimal $p$-envelope. Then it is shown that the Hamiltonian algebra $H(2; 1; \Phi(\tau))^{(1)}$ has a non-degenerate symmetric invariant bilinear form, and using this, the central extensions of $H(2; 1; \Phi(\tau))^{(1)}$ are determined. Moreover, some small irreducible representations of a Lie algebra, which modulo its solvable radical is isomorphic to the minimal $p$-envelope of $H(2; 1; \Phi(\tau))^{(1)}$, are classified. Under assumptions, which will be satisfied
in the classification proof, these results are applied to split off the solvable radical and derive some information on $p$-characters. The chapter concludes with studying certain properties of the Melikian algebras and their representations. It is shown that every Melikian algebra has a non-degenerate symmetric invariant bilinear form. As a consequence, every central extension of the smallest Melikian algebra $\mathcal{M}(1, 1)$ splits. Finally, a strong result on the $p$-characters of irreducible $\mathcal{M}(1, 1)$-modules of dimension at most 125, which will be needed later in the classification proof, is proved by using only elementary methods.

In Chapter 17 the author employs the known list of simple Lie algebras of absolute toral rank two, which was obtained in the second volume, to get more information on the trigonalizability of subalgebras and the $[p]$-nilpotency of certain distinguished elements. In turn, this yields a list of the possible 2-sections in a simple Lie algebra $L$ with respect to a torus $T$ of maximal dimension in its minimal $p$-envelope. The last three chapters are devoted to the following three cases, of which the first two need special methods: 1) There exists such a torus $T$ which is non-standard (i.e., the centralizer of $T$ in $L$ is not trigonalizable; this is only possible for $p = 5$, and then $L$ is isomorphic to a Melikian algebra), 2) there exists such a torus $T$ for which every 1-section is solvable (in this case one needs to consider 3-sections and simple Lie algebras of absolute toral rank three to prove that $L$ is a special Lie algebra or a Block algebra), and finally, 3) every torus of maximal dimension in the minimal $p$-envelope of $L$ is standard and has at least one non-solvable 1-section (in this case one needs the Mills-Seligman theorem and Kac’s recognition theorem for graded Lie algebras, which both (although without a proof) can be found in the first volume, to show that $L$ is of classical type or one of the remaining Cartan types).

As the first two volumes, the third volume is largely self-contained and only supposes that the reader is familiar with major parts of the book by the author and R. Farnsteiner [Modular Lie algebras and their representations. (English) Monographs and Textbooks in Pure and Applied Mathematics, 116. New York etc.: Marcel Dekker, Inc. viii, 301 p. (1988; Zbl. 0648.17003)], the first volume [Simple Lie algebras over fields of positive characteristic. I: Structure theory. (English) de Gruyter Expositions in Mathematics 38. Berlin: de Gruyter. viii, 540 p. (2004; Zbl. 1074.17005)], and the second volume [Simple Lie algebras over fields of positive characteristic. II: Classifying the absolute toral rank two case. (English) de Gruyter Expositions in Mathematics 42. Berlin: Walter de Gruyter. vi, 385 p. (2009; Zbl. 1190.17001)]. In conclusion, this book together with the two previous volumes will be very useful for researchers in modular Lie theory, and especially for those who want to understand the classification of finite-dimensional simple Lie algebras over an algebraically closed field of characteristic $p > 3$ from a uniform point of view.