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An invitation to quantum groups and duality. From Hopf algebras to multiplicative unitaries and beyond. (English)  

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Reviewer: Jörg Feldvoss (8086)

The aim of the book under review is to give an introduction to Hopf algebras and especially the duality of Hopf algebras in the setting of C*-algebras and von Neumann algebras. As Hopf algebras are generalizations of groups, in the latter setting Hopf algebras are generalizations of locally compact groups. A major motivation for considering Hopf algebras in the setting of operator algebras is the generalization of Pontrjagin duality to non-commutative locally compact groups. The book is divided into three parts which we are going describe now in some more detail.

The first part provides an introduction to Hopf algebras in a purely algebraic setting leading to the duality of algebraic quantum groups (i.e., multiplier Hopf *-algebras with positive left integrals or equivalently positive right integrals) as developed by Van Daele. As a preparation for the discussion of Hopf algebras in the setting of C*-algebras the last chapter of this part is devoted to compact algebraic quantum groups (i.e., Hopf *-algebras with positive left and positive right integrals) and a characterization of the latter in terms of their corepresentations. The concept of a compact algebraic quantum group is a common generalization of the group algebra of a finite or discrete group and of the Hopf algebra of representative functions on a compact group. Most of the results for compact algebraic quantum groups have analogues for cosemisimple Hopf algebras (i.e., ordinary Hopf algebras with a normalizable integral or equivalently with only semisimple comodules) but this is out of the scope of the book. This part concludes with a generalization of a classical result
saying that the Pontrjagin dual of a compact commutative group is discrete and vice versa to compact algebraic quantum groups. Here an algebraic quantum group is called discrete if as a $\ast$-algebra it is isomorphic to an algebraic direct product of full matrix $\ast$-algebras over the complex numbers. For this part only some basic knowledge of multilinear algebra is required.

The second part focuses on Hopf algebras in the setting of $C^*$-algebras and von Neumann algebras. After a discussion of the problems arising in the definition of a Hopf $C^*$-algebra or a Hopf von Neumann algebra, the author gives examples related to locally compact groups and lists the existing approaches to Hopf algebras in the setting of operator algebras. In the next two chapters Woronowicz’s theory of compact $C^*$-algebraic quantum groups together with several examples is presented which is particularly accessible and closely related to the theory of compact algebraic quantum groups. Multiplicative unitaries are fundamental to Hopf algebras in the setting of $C^*$-algebras and von Neumann algebras as well as to generalizations of classical Pontrjagin duality for commutative locally compact groups. An abstract definition and comprehensive discussion of multiplicative unitaries was given for the first time by Baaj and Skandalis and their theory is considered subsequently. This part ends with an overview of the theory of locally compact quantum groups as developed by Kustermans and Vaes which provides a comprehensive framework for the study of Hopf algebras in the setting of operator algebras and includes a generalization of the classical Pontrjagin duality for commutative locally compact groups that is valid for all locally compact groups. As illustrating examples the author sketches the construction of $E_{\mu}(2)$ and its dual as well as the self-dual quantum $az+b$ group which first were worked out by Woronowicz. The main goal of this last chapter of the second part is to give sufficient motivation for the existing theory of locally compact quantum groups and to explain the central analytic tools used therein but without providing any proofs.

The last part of the book is devoted to more special topics as coactions of quantum groups on $C^*$-algebras, reduced cross products, Kac systems culminating in Baaj-Skandalis duality which is a generalization of Takesaki-Takai duality for group actions (and the latter extends Pontrjagin duality for commutative locally compact groups to actions of such groups on $C^*$-algebras), pseudo-multiplicative unitaries on Hilbert spaces, and in the last chapter some results from the author’s dissertation on pseudo-multiplicative unitaries on $C^*$-modules and quantum groupoids in the setting of $C^*$-algebras. The second and the third parts assume some familiarity with certain definitions and results for Hilbert spaces, $C^*$-algebras, $C^*$-modules, and von Neumann algebras which for the convenience of the reader are summarized in an appendix.

The book is very well written and should be accessible and quite useful for graduate students and non-experts from other fields, especially because of its well chosen examples and a considerable effort to put the discussed material into perspective.