SYNOPTIC ABSTRACT

This paper considers the problem of simultaneous predicted response and prediction of average value of the study variable in a linear regression model when some prior exact restrictions are available, which bind the regression coefficients. The performance properties of the predictors based on restricted least squares and improved estimators are analyzed. A comparison of these predictors with respect to risk as the performance criterion is then presented.

Key Words and Phrases: linear regression model, exact linear restrictions, least squares estimator, restricted least squares estimator, improved estimators.
1. INTRODUCTION

Many authors have studied the predicted performance of predictors obtained by using different estimators for either the actual values or the average values of the study variable at a time, see, Singh and Dube (1995), Dube, Singh, V. (1996), Dube, M and V. Manocha (1998), Dube, M and V. Manocha (2002), Toutenburg and Trenkler (1990), Geiss (1993), etc. But in many situations there is a need to predict actual values and average values of the study variable simultaneously, for instance in a consumer market the life guarantee of an item on sale is of paramount importance. The manufacturer of the item claims the guarantee on the basis of its average life prediction, while the consumer takes decision regarding its purchase on the actual life prediction. The classical theory of prediction can predict either the manufacturer’s interest or the consumer’s interest at a time but can not predict both simultaneously. This can be achieved by defining a target function which considers both the average and actual values prediction of the study variable with possibly different weights assigned to them. This composite function would take care of the two predictions simultaneously and would allow us to desired weightage to them according to their respective importance in any given applications, see Zellner (1994), Shalabh (1995), Toutenburg and Shalabh (1996) and Shalabh and Toutenburg (2000).

In this paper, we have assumed a linear regression model with exact linear restrictions which binds the regression coefficients and study the properties of the predictors obtained by using ordinary least squares, restricted least squares and improved estimators proposed by Srivastava and Srivastava (1984). Section 2, describes the linear regression model and presents the target function for the predicted response and prediction of average values of the study variable. We also assume the availability of a set of exact linear restrictions on the regression coefficients and present the four predictors utilizing ordinary least squares, restricted least squares and improved estimators. The properties of these predictors are analyzed in the form of Theorems in section 3. Section 4 presents the comparison of these predictors. Finally, the derivation of the Theorems is presented in section 5.

2. MODEL SPECIFICATION AND TARGET FUNCTION

Let us postulate the following linear regression model
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\[ Y = X\beta + \sigma U \]  

(1)

where \( Y \) is an \((n \times 1)\) vector of \( n \) observations on the study variable; \( X \) is a \((n \times p)\) full column matrix of \( n \)-observations on \( p \) explanatory variables; \( \beta \) is a \((p \times 1)\) vector of regression coefficients and \( U \) is a \((n \times 1)\) vector of disturbances with mean zero, variance unity, measures of skewness \( \gamma_1 \) and measures of kurtosis \((\gamma_2 + 3)\).

If \( b \) denotes an estimator of \( \beta \), then the predictor for the values of study variable within the sample is generally formulated as \( \hat{T} = Xb \) which is used for predicting either actual values \( Y \) or the average values \( E(Y) = X\beta \) at a time. When the situation demands both the predicted response and prediction of average values of the study variable to gather, we may define the following target function:

\[ T(Y) = T = \alpha Y + (1-\alpha) E(Y) \]

(2)

and use \( \hat{T} = Xb \) for predicting the target function, where \( 0 \leq \alpha \leq 1 \) is a non-stochastic scalar specifying the weightage to be assigned to the predicted response and prediction of average values of the study variable; see, Shalabh (1995).

Let us suppose that we are given a set of \( q \) – exact linear restrictions binding the regression coefficients:

\[ r = R\beta \]

(3)

where \( r \) is a \((q \times 1)\) vector of known elements and \( R \) is a \((q \times p)\) full row rank matrix of known elements.

If the restriction (3) are ignored, then the ordinary least squares estimator of \( \beta \) is

\[ b_o = C^{-1}X'Y \quad ; \quad C = X'X \]

(4)

The estimator \( b_o \) defined in (4) does not, in general, obey the restrictions (3). This is avoided in restricted regression method of estimation, which yields the following restricted least squares estimator:

\[ b_R = b_o - C^{-1}R'(RC^{-1}R)^{-1}(Rb_o - r) \]

(5)

If we drop the property of linearity and unbiasedness, we can obtain improved estimator of \( \beta \) in a number of ways. For instance, we may consider Stein – rule estimators for the purpose:
where $k$ is the positive characterizing scalar.

Combining the philosophy behind restricted regression and stein-rule estimation, Srivastava and Srivastava (1984) proposed two families of improved estimators given by

$$b_{RS} = [b_R - k \frac{(Y - Xb_o)'(Y - Xb_o)}{b_o'Cb_o} A b_o]$$

(7)

$$b_{SR} = [I_p - k \frac{(Y - Xb_R)'(Y - Xb_R)}{b_R'Cb_R} A] b_R$$

(8)

where $b_{RS}$ and $b_{SR}$ denotes respectively the restricted stein-rule and stein-restricted least squares estimator of $\beta$.

$$A = [I - C^{-1}R'(RC^{-1}R)^{-1}R]$$

(9)

### 3. PREDICTION WITHIN THE SAMPLE

Employing (4), (5), (7) and (8), we get the following four predictors for the values of the study variable within the sample:

$$\hat{T}_o = Xb_o$$

(10)

$$\hat{T}_R = Xb_R$$

(11)

$$\hat{T}_{RS} = Xb_{RS}$$

(12)

$$\hat{T}_{SR} = Xb_{SR}$$

(13)

It is easy to see that the predictor (10) and (11) are unbiased with variance – covariance matrices

$$V(\hat{T}_o) = \sigma^2 \{ \lambda^2 I_n + (1 - 2\lambda)P_X \}$$

(14)

$$V(\hat{T}_R) = V(\hat{T}_o) - \sigma^2 (1 - 2\lambda)A^*$$

(15)

where

$$P_X = XC^{-1}X'$$

(16)

$$A^* = XC^{-1}R'(RC^{-1}R)^{-1}R'C^{-1}X'$$

(17)
Further, the risks associated with the predictors \( \hat{T}_o \) and \( \hat{T}_R \) is respectively given as

\[
\text{Risk}(\hat{T}_o) = \sigma^2 \{ n\hat{\lambda}^2 + (1-2\hat{\lambda})p \} \quad (18)
\]

\[
\text{Risk}(\hat{T}_R) = \text{Risk}(\hat{T}_o) - \sigma^2 (1-2\hat{\lambda})q \quad (19)
\]

In order to derive the properties of the predictors defined in (12) and (13), we shall use small sample asymptotic theory introduced by Kadane (1971) and define the following notations:

\[
M = I - P_X \quad (20)
\]

\[
M_j = [1 - \frac{j}{\beta^2 C\beta'} X\beta\beta' X'] : j=1,2,..... \quad (21)
\]

\[
M_{dj} = [1 - \frac{j}{\beta^2 C\beta'} X\beta\beta' A' X'] : j=1,2,..... \quad (22)
\]

\[
l_j = [1 - \frac{j}{\beta^2 C\beta'} g] : j=1,2,..... \quad (23)
\]

\[
g = \frac{1}{\beta^2 C\beta'} \beta' R'(RC^{-1}R')^{-1} R\beta : j=1,2,..... \quad (24)
\]

**Theorem 1:** When disturbances are small and not necessarily normal, the bias vector, mean squared error matrix and risk function associated with the predictor \( \hat{T}_{RS} \) are respectively given as

\[
\text{Bias}(\hat{T}_{RS}) = -\sigma^2 \frac{k(n-p)}{\beta^2 C\beta'} X\beta\beta' - \sigma^3 \frac{k\gamma_1}{\beta^2 C\beta'} (P_X - A^*) M_2 (I_n * M) e
\]

\[
+ \sigma^4 \frac{k}{[\beta^2 C\beta']^2} \left[ \gamma_2 \left\{ tr(M_4 - M) (I_n * M) X\beta \right\} + 2(n-p) \left( \frac{n}{2} - p - 1 \right) X\beta \right] \quad (25)
\]

where '*' denotes Hadamard product and 'e' is \((n \times 1)\) vector with all elements unity.

\[
M(\hat{T}_{RS}) = V(\hat{T}_R) - \sigma^3 \frac{k\gamma_1(1-\alpha)}{\beta^2 C\beta'} \left\{ \beta' A' X' (I_n * M) e + e' (I_n * M) X\beta \right\}
\]

\[
- \sigma^4 \frac{k}{\beta^2 C\beta'} \left[ \gamma_2 \left\{ (P_X - A^* - \alpha) (I_n * M) M_2 (P_X - A^*) \right\} \right]
\]
\[
+ \left( P_X - A^* \right) M_2 \left( P_X - A^* - \alpha \right) + k \text{tr} \left( I_n^* M \right) (l_1 - 1) \\
+ 2(n - p) \{ (l_1 - 1) \left( M_2 (P_X - A^*) - A^* (M_2 - A^*) \right) \}
\]

\[
+ \frac{k}{2} (n - p + 2) (l_1 - 1) \]  
(26)

Risk(\hat{T}_{RS}) = Risk(\hat{T}_R) - \sigma^2 \frac{2k\gamma_1}{\beta' C \beta} (1 - \alpha) \beta' A' X' (I_n^* M) e - \sigma^4 \frac{2k}{\beta' C \beta} \cdot \left[ \gamma_2 \{ \text{tr} \left( P_X - A^* \right) (I_n^* M) M_2 \left( P_X - A^* \right) + \frac{k}{2} \text{tr} M (I_n^* M) \right] (l_1 - 1) + (n - p) \{ (1 - \alpha) (p - q - 1 - l_4) \}
\]

\[
+ \frac{k}{2} (n - p + 2) (l_1 - 1) \]  
(27)

**Theorem 2:** When disturbances are small and not necessarily normal, the bias vector, mean squared error matrix and risk function associated with the predictor \( \hat{T}_{SR} \) are respectively given as

\[
\text{Bias}(\hat{T}_{SR}) = -\sigma^2 \frac{k(n - p + q)}{\beta' C \beta} XA\beta - \sigma^4 \frac{k\gamma_1}{\beta' C \beta} \left( P_X - A^* \right) M_{d_2}.
\]

\[
(I_n^* (M + A^*)) e + \sigma^4 \frac{k}{[\beta' C \beta]^2} \left[ \gamma_2 \{ \text{tr} \left( P_X - A^* \right) M_4 \right] (l_1 - 1) + (n - p + q) (p - q - 3 + l_4) \]  
X A\beta \]  
(28)

\[
M(\hat{T}_{SR}) = \nabla(\hat{T}_R) - \sigma^3 \frac{k\gamma_1}{\beta' C \beta} (1 - \alpha) \{ \beta' A' X' (I_n^* (M + A^*)) \} e
\]

\[
+ e' (I_n^* (M + A^*)) X A\beta \} - \sigma^4 \frac{k}{\beta' C \beta} \left[ \gamma_2 \left( \left( P_X - A^* - \alpha \right) \right) \cdot \left( I_n^* (M + A^*) \right) (P_X - A^*) M_{d_2} + M_{d_2} (P_X - A^*) (I_n^* (M + A^*)) \right] \)

\[
+ (n - p + q) \{ (1 - \alpha) \left( M_{d_2} (P_X - A^*) + (P_X - A^*) M_{d_2} \right)
\]

\[
+ k(n - p + q + 2) (l_1 - 1) \]  
(29)
\[ \text{Risk}(\hat{T}_{SR}) = \text{Risk}(\hat{T}_{R}) - \sigma^2 \frac{2k\gamma_1}{\beta C \beta} (1 - \alpha) \beta' A' X' (I_n * (M + A')) e \]

\[ -\sigma^4 \frac{2k}{\beta C \beta} \left[ \gamma_2 \{ \text{tr} \left( P_X - A^* - \alpha \right) (I_n * (M + A^*)) \left( P_X - A^* \right) M_{D2} \nonumber \right. \]

\[ + \frac{k}{2} \text{tr} \left\{ (M + A^*) (I_n * (M + A^*)) \right\} (l_1 - 1) \left. \right] + (n - p + q) \{ 1 - \alpha \} \]

\[ (p - q - 3 + l_2) + \frac{k}{2} (n - p + 2) (l_1 - 1) \]  

\[ (30) \]

4. A COMPARISON

On considering the risks associated with the predictors \( \hat{T}_o \) and \( \hat{T}_R \), we find that for average values prediction \( (\alpha = 0) \), the predictive risks of \( \hat{T}_o \) and \( \hat{T}_R \) are \( p\sigma^2 \) and \( (p - q)\sigma^2 \) respectively and predicted response \( (\alpha = 1) \), the predictive risks of \( \hat{T}_o \) and \( \hat{T}_R \) are \( (n - p)\sigma^2 \) and \( (n - p + q)\sigma^2 \) respectively. Thus, we observe that for predicted response, the predictive risk of \( \hat{T}_o \) is smaller than that of \( \hat{T}_R \) and just reverse holds for average values prediction.

On considering the risks associated with the predictors \( \hat{T}_{RS} \) and \( \hat{T}_{SR} \), we see that

\[ \text{Risk}(\hat{T}_R) - \text{Risk}(\hat{T}_{RS}) = \sigma^2 \frac{2k\gamma_1}{\beta C \beta} (1 - \alpha) \beta' A' X' (I_n * M) e + \sigma^4 \frac{2k}{\beta C \beta} \left[ \gamma_2 \{ \text{tr} \left( P_X - A^* - \alpha \right) (I_n * M) M_2 \left( P_X - A^* \right) \right. \]

\[ + \frac{k}{2} \text{tr} M (I_n * M) (l_1 - 1) \left. \right] + (n - p) \{ 1 - \alpha \} \]

\[ + (n - p) \{ 1 - \alpha \} (p - q - 1 - l_4) \]

\[ + \frac{k}{2} (n - p + 2) (l_1 - 1) \]  

\[ (31) \]

up to the order of our approximations \( o(\sigma^3) \), we observe that for predicted response \( (\alpha = 1) \), both the predictors \( \hat{T}_R \) and \( \hat{T}_{RS} \) have same risk and for average values prediction \( (\alpha = 0) \), the risk of \( \hat{T}_{RS} \) is smaller than the risk of \( \hat{T}_R \), if \( \gamma_1 \) and \( \beta \) have the
same sign. For normally distributed disturbances, we observe that up to the order of our approximations $o(\sigma^4)$, $\hat{T}_{RS}$ dominates $\hat{T}_R$ with respect to risk criterion, if $k$ satisfies,

$$0 < k < \frac{2(1-\alpha)}{(n-p+2)} \{ (p-q-2)g^{-1} + 4 \}$$ (32)

Further, for average values prediction ($\alpha=0$), $\hat{T}_{RS}$ still dominates $\hat{T}_R$ with respect to risk criterion and just reverse holds true for predicted response ($\alpha=1$).

* On considering the risks associated with the predictors $\hat{T}_R$ and $\hat{T}_{SR}$, we observe that

$$Risk(\hat{T}_R) - Risk(\hat{T}_{SR}) = \sigma^3 \frac{2k\gamma_1}{\beta'\beta} (1-\alpha) \beta' A' X' (I_n \ast (M + A^*)) e$$

$$+ \sigma^4 \frac{2k}{\beta'\beta} [ \gamma_2 \{ tr (P_X - A^* - \alpha) (I_n \ast (M + A^*)) \}$$

$$+ \{ (P_X - A^*) M_{D_2} + \frac{k}{2} tr \{ (M + A^*) (I_n \ast (M + A^*)) \} \}$$

$$+ (l_1 - 1) \} + (n-p+q) \{ (1-\alpha) (p-q-3+l_2)$$

$$+ \frac{k}{2} (n-p+q+2) (l_1 - 1) \} ]$$ (33)

up to the order of our approximations $o(\sigma^3)$, we observe that for predicted response ($\alpha=1$), both the predictors $\hat{T}_R$ and $\hat{T}_{SR}$ have same risk and for average values prediction ($\alpha=0$), the risk associated with $\hat{T}_{SR}$ is smaller than the risk associated with $\hat{T}_R$, if, $\gamma_1$ and $\beta$ have the same sign. For normally distributed disturbances, we observe that up to the order of our approximations $o(\sigma^4)$, $\hat{T}_{SR}$ dominates $\hat{T}_R$ with respect to risk criterion, if $k$ satisfies,

$$0 < k < \frac{2(1-\alpha)}{(n-p+q+2)} \{ (p-q-2)g^{-1} + 2 \}$$ (34)

Further, for average values prediction ($\alpha=0$), $\hat{T}_{SR}$ still dominates $\hat{T}_R$ with respect to risk criterion and just reverse holds true for predicted response ($\alpha=1$).

* On considering the risks associated with the predictors $\hat{T}_{RS}$ and $\hat{T}_{SR}$, we observe that
Risk(\(\hat{T}_{RS}\)) - Risk(\(\hat{T}_{SR}\)) = \sigma^3 \frac{2k\gamma_1}{\beta'C\beta} (1 - \alpha) \beta' A' X' (I_n * A') e + \sigma^4 \frac{2k}{\beta'C\beta}.

\[
\begin{align*}
\gamma_2 \{ tr \left(P_X - A^* - \alpha \right) \{ (I_n * (M + A^*)) M_{D2} & \right. \\
- (I_n * M) M_2 \} \{ P_X - A^* \} + \frac{k}{2} tr \left\{ (M + A^*) \right. & \\
\left. \left. \left\{ (I_n * (M + A^*)) - tr M (I_n * M) \right\} (l_1 - 1) \right\} \\
+ (n - p) \left\{ (1 - \alpha) . \left( \frac{q}{n - p} (p - q - 2) - l \left( \frac{2q}{n - p} \right) \right) & \\
+ \frac{q}{2} \left( l \left( \frac{2q + 2}{n - p} \right) - 1 \right) \right\} \right]\end{align*}
\]

(35)

up to the order of our approximations \(o(\sigma^3)\), we observe that for predicted response \((\alpha = 1)\), both the predictors \(\hat{T}_{RS}\) and \(\hat{T}_{SR}\) have same risk and for average values prediction \((\alpha = 0)\), the risk associated with \(\hat{T}_{SR}\) is smaller than the risk associated with \(\hat{T}_{RS}\), if, \(\gamma_1\) and \(\beta\) have the same sign. For normally distributed disturbances, we observe that up to the order of our approximations \(o(\sigma^4)\), \(\hat{T}_{SR}\) dominates \(\hat{T}_{RS}\) with respect to risk criterion, if \(k\) satisfies,

\[0 < k < \frac{2(1 - \alpha)}{q(n - p + \frac{q}{2} + 2)} \{ 3(n - p) + q - \frac{1}{2} (n - p - q(p - q - 2)) g^{-1} \} \] (36)

Further, for average values prediction \((\alpha = 0)\), \(\hat{T}_{SR}\) still dominates \(\hat{T}_{RS}\) with respect to risk criterion and just reverse holds true for predicted response \((\alpha = 1)\).

5. DERIVATIONS

Using (4) in (1), we obtain

\[Y - Xb_o = \sigma M U\] (37)

\[b_o = \beta + \sigma C^{-1} X' U\] (38)

So that up to order of our approximations \(o(\sigma^2)\)

\[b_o' C b_o = \beta' C \beta - 2 \sigma \beta' X' U + \sigma^2 U' P_s U\]
And for the same order of approximations

$$[b_o \cdot C_{b_o}]^{-1} = \frac{1}{\beta' C\beta} \left[ 1 - \frac{2\sigma}{\beta' C\beta} \beta' X' U - \sigma^2 \frac{1}{\beta' C\beta} \{ U'M_2 U - U'MU \} \right] \quad (39)$$

Using (37), (38) and (39) along with (5) and (7) in (12), we get

$$\hat{T}_{RS} - T = \sigma \phi_1 + \sigma^2 \phi_2 + \sigma^3 \phi_3 + \sigma^4 \phi_4$$

(40)

where

$$\phi_1 = (P_x - A^* - \alpha) U$$

(41)

$$\phi_2 = - \frac{k}{\beta' C\beta} U' MU \left( P_x - A^* \right) X\beta$$

(42)

$$\phi_3 = - \frac{k}{\beta' C\beta} U' MU \left( P_x - A^* \right) M_2 U$$

(43)

$$\phi_4 = \frac{k}{[\beta' C\beta]^2} U' MU \left[ (U'M_2 U - U'MU) X\alpha \beta \right] + 2(P_x - A^*) UU' X\beta \]$$

(44)

Here, it is easy to see that

$$E(\phi_1) = 0$$

(45)

$$E(\phi_2) = - \frac{k(n - p)}{\beta' C\beta} X\alpha \beta$$

(46)

$$E(\phi_3) = - \frac{k\gamma_1}{\beta' C\beta} \left( P_x - A^* \right) M_2 (I_n \ast M)e$$

(47)

$$E(\phi_4) = \frac{k}{[\beta' C\beta]^2} \left[ \gamma_2 \left\{ tr(M_2 - M) \left( I_n \ast M \right) X\alpha \beta \right\} + 2(P_x - A^*) \right] (I_n \ast M) X\beta \} + 2(n - p) \left( \frac{n}{2} - p - 1 \right) X\alpha \beta \]$$

(48)

Utilizing (45), (46), (47) and (48) in (40), we obtain the expression (25) of the Theorem1.

$$[\hat{T}_{RS} - T] \left[ \hat{T}_{RS} - T \right]' = \sigma^2 \phi_1 \phi_1' + \sigma^3 (\phi_2 \phi_2' + \phi_2' \phi_2) + \sigma^4 (\phi_3 \phi_3' + \phi_3' \phi_3')$$

(49)

Here,

$$E[\phi_1 \phi_1'] = \{ \lambda^2 I_n + (1 - 2\lambda) P_x \}$$

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\[ E[\phi_1, \phi_2'] = -\frac{k_1(1-\alpha)}{\beta C\beta} \beta' A'X' (I_n * M)e \] (51)

\[ E[\phi_3'] = -\frac{k}{\beta C\beta} \left[ \gamma_2 \left( P_x - A' \right) \right] \left( I_n * M \right) M_z \left( P_x - A' \right) \]

\[ + (n-p) \left( 1-\alpha \right) M_z \left( P_x - A' \right) - A' \left( M_z - A' \right) \right] \] (52)

\[ E[\phi_2'] = -\frac{k^2}{\beta C\beta} \left[ \gamma_2 \left( M_n * M \right) + (n-p)(n-p+2) \right] g \] (53)

Utilizing (50), (51), (52) and (53) in (49), we obtain the expression (26) of Theorem 1.

\[ \text{Risk} \left( \hat{T}_{RS} \right) = E \left[ \left( \hat{T}_{RS} - T \right) \left( \hat{T}_{RS} - T \right)' \right] \]

\[ = \text{tr} E \left[ \left( \hat{T}_{RS} - T \right) \left( \hat{T}_{RS} - T \right)' \right] \] (54)

Similarly, utilizing (54), the expression (27) of Theorem 1 can be obtained.

Using (5) in (1), we obtain

\[ Y - Xb_R = \sigma(M + A')U \] (55)

\[ b_R = \beta + \sigma AC^{-1}X'U \] (56)

So that up to order of our approximations \( o(\sigma^2) \)

\[ b_R'Cb_R = \beta' C\beta - 2\sigma \beta' A'X'U + \sigma^2U'(P_x - A')U \]

and for the same order of approximations

\[ [b_R'Cb_R]^{-1} = \frac{1}{\beta' C\beta} \left[ 1 - \frac{2\sigma}{\beta' C\beta} \beta' A'X'U - \sigma^2 \right] \frac{1}{\beta' C\beta} \left[ U'(P_x - A')M_4 \right. \]

\[ \left. (P_x - A')U \right] \] (57)

Using (55), (56) and (57) along with (5) and (8) in (13), we get

\[ \hat{T}_{SR} - T = \sigma \eta_1 + \sigma^2 \eta_2 + \sigma^3 \eta_3 + \sigma^4 \eta_4 \] (58)

where

\[ \eta_1 = (P_x - A' - \alpha)U \] (59)

\[ \eta_2 = -\frac{k}{\beta' C\beta} U'(M + A')U XA\beta \] (60)

\[ \eta_3 = -\frac{k}{\beta' C\beta} U'(M + A')U (P_x - A') M_{D_2}U \] (61)

\[ \eta_4 = \frac{k}{(\beta' C\beta)^2} U'(M + A')U \left\{ U'(P_x - A')M_4 (P_x - A')U \right\} \]
\[ + 2\left(P_x - A^*\right)UU' \} XA\beta \quad (62) \]

Here, it is easy to see that

\[ E(\eta_1) = 0 \quad (63) \]

\[ E(\eta_2) = -\frac{k(n-p+q)}{\beta C\beta} XA\beta \quad (64) \]

\[ E(\eta_3) = -\frac{k\gamma_1}{\beta C\beta} \left(P_x - A^*\right)M_{d2}(I_n^* (M + A^*))e \quad (65) \]

\[ E(\eta_4) = \frac{k}{[\beta C\beta]^2} \left[ \gamma_2 \left\{ tr(P_x - A^*)M_4 (P_x - A^*) (I_n^* (M + A^*)) \right. \right. \]
\[ \left. \left. + 2\left(P_x - A^*\right) (I_n^* (M + A^*)) \} XA\beta \right. \right. \]
\[ \left. \left. + 2(n-p) \left( \frac{n}{2} - p - 1 \right) XA\beta \right. \right. \] \quad (66)

Utilizing (63), (64), (65) and (66) in (58), we obtain the expression (28) of Theorem 2.

Further

\[ \left[ \hat{T}_{SR} - T \right] \left[ \hat{T}_{SR} - T \right]' = \sigma^2 \eta, \eta_1' + \sigma^3 (\eta, \eta_2', + \eta, \eta_2') \]
\[ + \sigma^4 (\eta, \eta_3', + \eta, \eta_3', + \eta, \eta_3') \quad (67) \]

Here,

\[ E[\eta, \eta_1'] = \left\{ \lambda I_n^* + (1-2\lambda)(P_x - A^*) \right\} \quad (68) \]

\[ E[\eta, \eta_2'] = -\frac{k\gamma_1(1-\alpha)}{\beta C\beta} \beta' A' X' (I_n^* (M + A^*))e \quad (69) \]

\[ E[\eta, \eta_3'] = -\frac{k}{\beta C\beta} \left[ \gamma_2 \left(P_x - A^* - \alpha \right) (I_n^* (M + A^*)) \left(P_x - A^*\right) M_{d2} \right. \]
\[ \left. \left. + (n-p+q) \left( 1-\alpha \right)(P_x - A^*)M_{d2} \right] \quad (70) \]

\[ E[\eta_2, \eta_2'] = \frac{k^2}{\beta C\beta} \left\{ \gamma_2 tr(M + A^*) (I_n^* (M + A^*)) \right. \]
\[ \left. \left. + (n-p+q)(n-p+q+2) \right\} \quad g \quad (71) \]

Utilizing (68), (69), (70) and (71) in (67), we obtain the expression (29) of Theorem 2.

Similarly,

\[ Risk(\hat{T}_{SR}) = trE \left[ (\hat{T}_{SR} - T) (\hat{T}_{SR} - T)' \right] \quad (72) \]

Thus, utilizing (72), we obtain the expression (30) of Theorem 2.
REFERENCES


