Families of proper generalized neighbor designs

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Abstract

A generalized neighbor design relaxes the equality condition on the number of times two treatments occur as neighbors in the design. In this article we have constructed a new series of generalized neighbor designs with equal block sizes, a series of neighbor designs of Rees [1967. Some designs of use in serology. Biometrics 23, 779–791] and a series of neighbor designs with two distinct block sizes. Two more new series of GN2 designs are also constructed for even number of treatments. It has been shown that quasi neighbor designs introduced by Preece [1994. Balanced Ouchterlony neighbor designs. J. Combin. Math. Combin. Comput. 15, 197–219] are special cases of generalized neighbor designs with \( t = 2 \). All the designs given here are binary. A new definition—partially balanced circuit design is introduced which is a special case of generalized neighbor designs with binary blocks. © 2006 Elsevier B.V. All rights reserved.

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1. Introduction

A block design is called proper if all its blocks are of equal size. Rees (1967) introduced the “neighbor designs” for use in serological experiment. According to these designs many virus preparations were arranged in circular plates so that every preparation appears as a neighbor of every other preparation equally often.

Rees (1967) definition of neighbor designs: There are \( v \) types of virus preparations (treatments) to be arranged in \( b \) circular plates containing \( k \) treatments. Each treatment appears \( r \) times in the design (not necessarily on \( r \) distinct blocks) and is a neighbor of every other treatment exactly \( \lambda \) times. The following is an example of \( v = b = 9 \), \( k = r = 4 \), \( \lambda = 1 \) design.

\[
B1 = (1, 2, 9, 6), \quad B2 = (2, 3, 1, 7), \quad B3 = (3, 4, 2, 8), \\
B4 = (4, 5, 3, 9), \quad B5 = (5, 6, 4, 1), \quad B6 = (6, 7, 5, 2), \\
B7 = (7, 8, 6, 3), \quad B8 = (8, 9, 7, 4), \quad B9 = (9, 1, 8, 5).
\]

Rees (1967) constructed such designs with odd number of treatments, i.e., \( v = 2n + 1 \) and \( \lambda = 1 \), with complete and incomplete block designs. Consequently, Lawless (1971) gave constructions of incomplete block neighbor designs with \( \lambda \) being constant. Hwang (1973), Hwang and Lin (1977), Das and Saha (1976), and Kageyama (1979) gave some more constructions of incomplete block neighbor designs with higher values of \( \lambda \).

As we have seen in the definition of Rees neighbor designs, the blocks may or may not be binary. The binary neighbor designs were given the name “balanced circuit designs” by Rosa and Huang (1975) putting a restriction on
the block sizes that \( k \leq v \). They defined a balanced circuit design as: A balanced circuit design with parameters \( v, b, r, k, \lambda (\text{briefly, } BC\text{(v, k, } \lambda)) \) is an arrangement of \( v \) elements into \( b \) \( C \)-blocks such that each \( C \)-block contains \( k \) elements, each element occurs in exactly \( rC \)-blocks and any two distinct elements are linked in exactly \( \lambda C \)-blocks. These designs are studied by Rosa and Huang (1975), Hwang and Lin (1977) and Bermond et al. (1978). Bailey et al. (2003) constructed balanced circuit designs with \( v = k \). In the literature construction of such complete block neighbor designs are also known as children’s round dance problem.

Subsequent to Rees (1967), several researchers have defined the “neighbor designs” in different ways. For example, Preece (1994), defines “balanced Ouchterlony neighbor designs (BONDS)” as follows:

A BOND is an arrangement where the members of a set \( S \) of \( v \) distinct elements are disposed in \( b \) blocks so that

(i) each block contains \( k \) elements \((k > 2)\) that are drawn from \( S \) but are not necessarily all distinct;
(ii) the elements in each block are arranged on the circumference of a circle so that each of these elements has two neighbors;
(iii) each member of \( S \) appears exactly \( r \) times throughout the arrangement;
(iv) no element of \( S \) ever has itself as a neighbor;
(v) every element of \( S \) has each member of \( S \) as a neighbor exactly \( \lambda' \) times.

This definition coincides with the definition of neighbor designs given by Rees (1967). This generalizes the Rees neighbor designs by relaxing some of the conditions on the parameters. According to Preece (1994), a quasi Rees neighbor design is one that satisfies conditions (i)–(iv) of BOND and modified condition (v) as:

\[ (v^*) \text{ every element from } S \text{ has each other element as a neighbor exactly once, except that it has just one of the other elements as a neighbor exactly twice.} \]

This modified definition would give the designs with two types of \( \lambda \)-values. Namely \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \) and for a given treatment, there is exactly one treatment which occurs twice as neighbors.

Prior to definition of “quasi Rees neighbor designs”, given by Preece (1994), Misra et al. (1991) introduced the definition of “generalized neighbor designs”. In this definition they relaxed the equality conditions on \( \lambda \) as well as on \( k \). As per this definition, for a fixed treatment, the rest of \((v - 1)\) treatments can be divided into groups according to values of \( \lambda \) independent of the treatment. A design with \( t \) such groups was given name GNt-design. This property of generalized neighbor designs is similar to that of partially balanced block designs. Following the definition of balanced circuit designs in Rosa and Huang (1975) we can define partially balanced circuit designs as the special class of GNt-designs with \( k < v \) and all blocks binary. (The blocks are binary when a treatment occurs in a block atmost once.) Thus whenever the GNt-designs are proper, and binary they can be equivalently named as partially balanced circuit designs. The GNt-designs constructed in here are all binary block designs hence can be called as “partially balanced circuit designs”.

**Definition.** A proper and binary GN1-design with equal replication numbers is a partially balanced circuit design with parameters \( v, b, r, k, \lambda_1, \lambda_2, \ldots, \lambda_t, n_1, n_2, \ldots, n_{t-1} \).

Quasi Rees neighbor designs of Preece (1994) are GN2-designs of Misra et al. (1991) with \( \lambda_1 = 2, \lambda_2 = 1 \) and \( n_1 = 1 \). Later on Chaure and Misra (1996) gave some more methods of construction of generalized neighbor designs.

In Misra et al. (1991), we have constructed a series of GN3-designs, and two series of GN2-designs with unequal block sizes and with odd number of treatments whereas in the present article proper GNt-designs are constructed. Also a new series of GN2 designs with even number of treatments is also given.

**2. Definitions and notations**

**Definition of generalized \( t \)-neighbor designs in Misra et al. (1991):**
An arrangement of \( v \) treatments in \( b \) circular blocks such that

(i) each treatment appears \( r \) times in the design (not necessarily in \( r \) distinct blocks);
(ii) blocks have \( k_1, k_2, \ldots, k_b \) treatments (same treatment should not occur sided by side);
any two treatments can occur as neighbor $\lambda_1, \lambda_2, \ldots, \lambda_t$ times;

for a given treatment $\theta$, there are $n_1$ treatments which occur $\lambda_i$ times as neighbor where the number $n_1$ is independent of the treatment $\theta$.

The notations used for such designs are $\text{GN}_t$-design, where $t$ denotes the number of distinct values of $\lambda$. The $\text{GN}_t$-designs have following two properties:

1. $\sum n_1 = v - 1$.
2. $\sum n_1 \lambda_1 = 2r$.

When $k_i = k$ for every $i$, and $\lambda_1 = \lambda_2 = \cdots = \lambda_t = \lambda$, we get neighbor designs of Rees (1967).

Let $s$ and $r$ be either prime or prime power, and $\text{GF}(s)$ and $\text{GF}(t)$ be corresponding Galois fields with primitive elements, respectively, $x$ and $y$. Let $\text{GD}(v)$ for $v = s \ast t$ denote the Galois domain with $s \ast t$ elements as listed below:

$$0 = (0,0), \quad w^m = (x^m,0), \quad u^j = (0,y^j), \quad z^{ij} = (x^i,y^j)$$

3. Construction principle

This construction principle is taken from Misra et al. (1991). Let $v$ treatments of a design come from a module $M$. That is the treatments are given name from the set $\{0,1,2,\ldots,v\}$. If $(i_1,i_2,\ldots,i_k)$ is a block of size $k$ then $F_k = (i_2 - i_1, i_3 - i_2, \ldots, i_1 - i_k)$ and $B_k = (i_1 - i_2, i_2 - i_3, \ldots, i_k - i_1)$. Define the forward and backward differences with $F_k = -B_k$.

Consider $s$ initial blocks

$$I_1 = (i_1^1, i_1^2, \ldots, i_k^1), \quad I_2 = (i_1^2, i_2^1, \ldots, i_k^2), \quad I_3 = (i_1^3, i_2^3, \ldots, i_k^3), \quad \cdots \quad I_s = (i_1^s, i_2^s, \ldots, i_k^s),$$

$I_1, I_2, \ldots, I_s$ when developed modulo $v$, generate a $\text{GN}_t$-design whenever the following two conditions are satisfied.

1. The sum of forward and backward differences arising from each of these initial blocks is zero.
2. Among the totality of forward and backward differences arising from these $s$ blocks, there are $n_1$, non-zero elements of $M$ which occur $\lambda_i$ times, $i = 1,2,\ldots,t$.

Constructions in the following Section 4 are all based on this principle.

4. Results

**Theorem 1.** Let $t = 3$ and $s > 3$ such that $s$ is an odd prime or odd prime power. Then $(s - 1, t - 1) = 2$, and let $v = 3s$.
Consider the initial blocks:

$$I_1 = \{(1,1), z^{11}, z^{22}, \ldots, z^{(h-1)(h-1)}\}$$

and

$$I_2 = \{w^0, w^1, \ldots, w^{(s-2)}\},$$

where $h = ((s-1) \ast (t-1))/2 = s - 1$.

When developed modulo $v$, yield a series of $\text{GN}_3$-designs with parameters $v = 3s$, $b = 6s$, $k = s - 1, r = 2(s - 1)$, $n_1 = 2(s - 1), n_2 = s - 1, n_3 = 2$, with $\lambda_1 = 1, \lambda_2 = 2$ and $\lambda_3 = 0$.

**Proof.** The $v$ blocks of size $(s - 1)$ obtained from $I_1$ have forward and backward differences of the form $\pm(z^{i+1} - z^{i-1})$. These differences can be written in terms of $x$ and $y$ as $\pm(x^i(x - 1), y^i(y - 1))$ which can be expressed as $\pm(x^i y^j - x^j y^i)$ for a fixed $p$ and fixed $q$. That is for a given treatment all such differences appear exactly once so that $n_1 = 2(s - 1)$ and $\lambda_1 = 1$. Now we develop $I_2$ modulo $v$ to give $v$ blocks. The forward and backward differences can be expressed as powers of $w^j$ each occurring twice, once as forward and once as backward difference. Thus counting $n_2 = s - 1$ and $\lambda_2 = 2$. The remaining two non-zero elements of $\text{GD}(v)$ do not occur together in any of the blocks giving $\lambda_3 = 0$ with $n_3 = 2$. □
According to above mapping initial block

Theorem 2.

Table 2

Rees neighbor designs with equal block size

<table>
<thead>
<tr>
<th>s = t</th>
<th>v</th>
<th>b</th>
<th>k</th>
<th>n1</th>
<th>n2</th>
<th>n3</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>25</td>
<td>150</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>49</td>
<td>392</td>
<td>6</td>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note 1: If we take \( s = 3 \) in Theorem 1, all designs are with \( k = 2 \) and consequently such designs are simpler from statistical view point.

Note 2: The series of neighbor designs given by Theorem 1 is different from that of Rees in its block size, hence different. For example, with \( v = 21 \) treatments, Rees gives solutions for designs with \( k = 21, 10, 7 \) and 5 whereas our series gives a design with \( k = 6 \).

Table 1 lists the parametric values of some of the GN3-designs obtained from Theorem 1.

Note 3: All designs in the series in Theorem 1 have binary blocks thus they can be alternatively called as partially balanced circuit designs similar to circuit designs of following is an example of the construction of second design given in the table.

Example 1. \( s = 7, t = 3, v = 21, b = 42, r = 12, k = 6, n_1 = 12, n_2 = 4, n_3 = 2, \) with \( \lambda_1 = 1, \lambda_2 = 2 \) and \( \lambda_3 = 0, h = 6 \). Primitive element of GF(7) is \( x = 3 \) and primitive element of GF(3) is \( y = 2 \).

Thus \( I_1 : \{(1, 1), (3, 2), (2, 1), (6, 2), (4, 1), (5, 2)\} \) and \( I_2 : \{(1, 0), (3, 0), (2, 0), (6, 0), (4, 0), (5, 0)\} \).

The 21 treatments and their mapping are listed as follows:

\[
\begin{align*}
(0, 0) & \rightarrow 1 \\
(0, 1) & \rightarrow 8 \\
(0, 2) & \rightarrow 15 \\
(1, 0) & \rightarrow 2 \\
(1, 1) & \rightarrow 9 \\
(1, 2) & \rightarrow 16 \\
(2, 0) & \rightarrow 3 \\
(2, 1) & \rightarrow 10 \\
(2, 2) & \rightarrow 17 \\
& \text{...}
\end{align*}
\]

According to above mapping initial block \( I_1 : \{9, 18, 10, 21, 12, 20\} \) and \( I_2 : \{2, 4, 3, 7, 5, 6\} \). These two initial blocks when developed modulo 21 would give the 42 blocks of our designs.

Corollary 1. In Theorem 1 if we take \( s = 3 \) and assign different odd prime values to \( t \), then we get a series of GN3-designs with two different block sizes with \( k_1 = t - 1 \) and \( k_2 = 2, \) thus half of the blocks in the design are of size 2 each. Some of the blocks are of size 2 and \( v \) blocks are of size greater than 2. Therefore, the designs obtainable from Corollary 1 are of some statistical interest.

Theorem 2. Let \( v = s \times t \), \( s \) and \( t \) both are either odd primes or power of odd primes such that \( s = t \) then \( (s - 1, t - 1) = s - 1 \) then \( (s - 1) \) cosets of multiplicative subgroup \( \{(1, 1), z^{11}, z^{22}, \ldots, z^{(k-1)(h-1)}\} \) of GD(v) taken as initial blocks, \( I_1, I_2, \ldots, I_{(s-1)} \) along with two more initial blocks \( I_{d+1} = \{w^0, w^1, \ldots, w^{(s-2)}\} \) and \( I_{d+2} = \{u^0, u^1, \ldots, u^{(t-2)}\} \) when developed modulo \( v \), gives a series of neighbor designs with \( \lambda = 2 \) and other parameters as \( v = s^2, b = (s + 1) \times s^2, k = s - 1, r = s^2 - 1. \) Here \( h = (s - 1) \times ((t - 1)/(s - 1) = s - 1) \).

Table 2 lists the parameters of two designs which can be generated using Theorem 2.

The designs thus generated can also be categorized as balanced circuit designs.
All these six initial blocks when developed modulo 25 give 150 blocks which give the neighbor design.

Let Theorem 3.

Table 3

<table>
<thead>
<tr>
<th>s</th>
<th>t</th>
<th>v</th>
<th>b</th>
<th>(k_1)</th>
<th>(k_2)</th>
<th>(r)</th>
</tr>
</thead>
<tbody>
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<td>180</td>
<td>8</td>
<td>4</td>
<td>44</td>
</tr>
<tr>
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<td>13</td>
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<td>64</td>
</tr>
<tr>
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<td>340</td>
<td>16</td>
<td>4</td>
<td>84</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>125</td>
<td>500</td>
<td>24</td>
<td>4</td>
<td>124</td>
</tr>
<tr>
<td>5</td>
<td>29</td>
<td>145</td>
<td>580</td>
<td>28</td>
<td>4</td>
<td>144</td>
</tr>
<tr>
<td>5</td>
<td>37</td>
<td>185</td>
<td>740</td>
<td>36</td>
<td>4</td>
<td>184</td>
</tr>
</tbody>
</table>

Note: Note the large values of \(r\). In the olden days there was a restriction on the size of experiment due to limitations on the availability of experimental material as well as the computing facilities. Nowadays the designs being applied to large experiments (such as micro arrays,) and with the advance in the computing power, its not difficult to find applications of such designs.

Example 2. For \(s = t = 5\), \(v = 25\), \(b = 150\), \(k = 4\), \(r = 24\), \(h = 4\). \(GF(s) = GF(t) = GF(5)\), primitive element of \(GF(5)\) is 2. \(GD(25)\) has a multiplicative subgroup \(\{1, 1\}(2, 2), (4, 4), (3, 3)\). The four cosets of this multiplicative subgroup are obtained by multiplying it by \((1, 1), (2, 1), (3, 1)\) and \((4, 1)\), respectively.

\[ I_1 : \{(1, 1), (2, 2), (4, 4), (3, 3)\}, \quad I_2 : \{(2, 1), (4, 2), (3, 4), (1, 3)\}, \]
\[ I_3 : \{(3, 1), (1, 2), (2, 4), (4, 3)\}, \quad I_4 : \{(4, 1), (3, 2), (1, 4), (2, 3)\}. \]

The initial blocks \(I_1, I_2, I_3, I_4\) are the four cosets. The other two initial blocks are as follows:

\[ I_5 : \{(0, 1), (0, 2), (0, 3), (0, 4)\}, \quad I_6 : \{(1, 0), (2, 0), (3, 0), (4, 0)\}. \]

All these six initial blocks when developed modulo 25 give 150 blocks which give the neighbor design.

Corollary 2. If \(t\) and \(s\) are odd prime or prime powers such that \((s - 1, t - 1) = s - 1\) or \((s - 1, t - 1) = t - 1\) then we get a series of Rees neighbor designs with two different block sizes with parameters:

\[ v = s * t, \quad b = (s + 1) * s * t, \quad k_1 = t - 1, \quad k_2 = s - 1, \quad r = s * t - 1 \quad \text{and} \quad h = 2. \]

Proof. Let \((s - 1, t - 1) = s - 1\). Consider the following initial blocks generated from the multiplicative subgroup of \(GD(v)\). The size of this multiplicative subgroup is \((s - 1)\). Thus there will be \((s - 1)\) cosets of size \((s - 1)\) each giving \((s - 1)\) initial blocks of size \((s - 1)\). On the other hand, one more initial block of size \((s - 1)\) is of the form \({w^n, w^1, \ldots, w^{s-2}}\) and the last initial block is of the form \({u^0, u^1, \ldots, u^{s-2}}\).

Thus finally we have \(s\) initial blocks of size \((s - 1)\) and one initial block of size \((t - 1)\) generating \(v * s\) blocks of size \((s - 1)\) and \(v\) blocks of size \((t - 1)\), respectively. After developing all these initial blocks modulo \(v\) we get the above neighbor design. \(\square\)

Table 3 lists the parameters of the designs with two types of block sizes as given in Corollary 2.

| Neighbor designs with \(v = 2m\). |

Misra et al. (1991) constructed \(GN_t\) designs only for odd number of treatments. In the following theorem we construct a series of \(GN_t\) designs with even number of treatments with certain restrictions.

Theorem 3. Let \(v = st\), with \(s = 3\) and \(t\) is an even prime power. Then the initial blocks

\[ I_1 = \{(1, 1), z^{11}, z^{22}, \ldots, z^{(h-1)(h-1)}\} \]

and

\[ I_2 = \{w^0, w^1, \ldots, w^{(s-2)}\}, \]

where \(h = ((s - 1) * (t - 1))/2 = t - 1\) when developed mod \(v\), yields a series of \(GN_2\) designs with \(v = 3t\), \(b = 2v\), \(r = h + 2\), \(k_1 = h\), \(k_2 = 2\), \(\lambda_1 = 1\), \(\lambda_2 = 0\), \(n_1 = 2h + 2\).
Example 3. For \( s = 3 \) and \( t = 4 \), the GN2 design has \( v = 12 \), \( b = 24 \), \( h = 3 \), \( r = 5 \), \( k_1 = 3 \), \( k_2 = 2 \), \( \lambda_1 = 1 \), \( \lambda_2 = 0 \), \( n_1 = 8 \).

The following theorem gives a series of proper GN2 designs with complete binary blocks.

**Theorem 4.** With \( v = 2^m \) treatments, generate a base block as follows:

\[
B = \{0, v - 1, 1, v - 2, 2, \ldots, v/2 - 1, v/2\}.
\]

This base block when developed mod \( v \) gives GN2 design with the parameters \( v = b = k = r = 2^m \), \( \lambda_1 = 2 \), \( \lambda_2 = 4 \), \( n_1 = v - 2 \).

Further if the treatments are labeled as in the base block, then the pairs which occur as neighbor four times are \((i, i + v/2)\) for \( i = 0, 1, \ldots, v/2\).

Example 4. For \( v = 6 \), the base block is \((0, 5, 1, 4, 2, 3)\).

5. Discussion

We have constructed four new series of equi block sized neighbor designs, and a series of Rees neighbor designs with two different block sizes. Our first series of equi block designs from Theorem 1 is a GN3-design which can also be called as partially balanced circuit designs, the second series of designs from Theorem 2 is a series of Rees neighbor designs with \( \lambda = 2 \). This can also be named as series of balanced circuit designs. A corollary to Theorems 1 and 2 gives Rees neighbor designs with two distinct block sizes. The last two series of designs constructed from Theorems 3 and 4, are GN2 with designs for even number of treatments. Atleast two questions are open in this article, first is the construction of partially balanced circuit designs and second is construction of GN2 designs with even number of treatments in general.

As it is clear from all the constructions given in this article that balance has been observed only in terms of neighbors. That is how many times two treatments occur together side by side in the design. Thus these neighbor designs are not necessarily variance balanced. The main objective is to study the neighboring effects along with the main effects of the treatments. And hence need a separate analysis technique then the usual block designs. The author is working on statistical properties and the analysis of neighbor designs which are different from the two way block designs.

References