PREDICTIVE PERFORMANCE OF THE IMPROVED ESTIMATORS UNDER STOCHASTIC RESTRICTIONS IN LINEAR REGRESSION MODELS.

Manoj Kumar,
Department of Statistics, Punjab University,
Chandigarh – 160014, INDIA
E – Mail: man_tiwa@yahoo.com

Nutan Mishra,
Department of Mathematics and Statistics,
University of South Alabama, Mobile, AL–36688, USA.

Rajinder Gupta
Department of Statistics, University of Jammu,
J&K – 180006, INDIA.

Abstract

This paper considers the problem of simultaneous prediction of actual and average values of the study variable in a linear regression model with some stochastic linear restrictions. The performance properties of the predictors obtained by employing ordinary least squares and improved estimators are analyzed. A comparison among these predictors with respect to risk as the performance criterion is then presented.

Key Words:– Linear regression model, Stochastic linear restrictions, Target function, Ordinary Least squares estimator, Mixed regression estimator, Improved estimators.
1 Introduction

Generally predictions from a linear regression model are made either for the actual values of the study variable or for the average values of the study variable at a time, see e.g., Toutenburg and Trenkler (1990) etc. However, situations may occur in which one may be required to consider the predictions of both the actual and average values of the study variable simultaneously. For the purpose Shalabh (1995) has proposed a composite target function which considers the prediction of both the actual and average values of the study variable together.

In this paper, we have assumed a linear regression model with stochastic linear restrictions and study the properties of the predictors obtained by using ordinary least squares estimator, mixed regression estimator and improved estimators proposed by Srivastava and Srivastava (1983). Section 2, of this paper deals with model specifications and presents target function which considers both the actual values and average values of the study variable. In this section we also assumes the availability of stochastic linear restrictions on the regression coefficients and present four predictors obtained by employing ordinary least squares estimator, mixed regression estimator and improved estimators. In section 3, we present the properties of the predictors in the form of theorems. A comparative study of these predictors on the basis of their performance properties contained in section 4. Finally, section 5, deals with the derivations of theorems presented in section 3.

2 Model Specifications and Target Function

Let us postulate the following linear regression model

\( Y = X\beta + \sigma \omega \)  

where \( Y \) is an \((n \times 1)\) vector of \( n \) - observations on the study variable; \( X \) is a \((n \times p)\) full column matrix of \( n \) - observations on \( p \) - explanatory variables; \( \beta \) is a \((p \times 1)\) column vector of regression coefficients and \( \sigma \) is an unknown positive scalar and \( \omega \) is an \((n \times 1)\) vector of disturbances with mean zero, variance unity, measures of skewness \( \gamma_1 \) and measures of kurtosis \( \gamma_2 + 3 \).
If \( b \) denotes an estimator of \( \beta \), then the predictor for the values of study variable within the sample is generally formulated as \( \hat{T} = Xb \) which is used for predicting either actual values \( Y \) or the average values \( E(Y) = X\beta \) at a time. When the situation demands prediction for both the actual and average values of the study variable together, we may define the following target function:

\[
T(Y) = T = \alpha Y + (1-\alpha) E(Y)
\]

and use \( \hat{T} = Xb \) for predicting the target function, where \( 0 \leq \alpha \leq 1 \) is a non-stochastic scalar specifying the weightage to be assigned to the prediction of actual and average values of the study variable; see, [Shalabh (1995)].

Let us suppose that we have given a set of stochastic linear restrictions on the regression coefficients which are

\[
q = Q\beta + V
\]

where \( q \) is a \((J \times 1)\) vector of known elements, \( Q \) is a \((J \times p)\) full row rank matrix of known elements and \( V \) is a \((J \times 1)\) random vector with mean zero and variance-covariance matrix \( \Omega \), which is positive definite and known.

Further, it is assumed that \( \omega \) and \( V \) are stochastically independent. If the restrictions given in (2.3) are completely ignored, then the ordinary least squares estimator of \( \beta \) is

\[
b_o = C^{-1}X'Y; \quad C = X'X
\]

When the restrictions (2.3) are incorporated, we get the mixed regression estimator, see, Theil and Goldberger (1961) and Theil (1963, 1971).

\[
b_m = \left[ C + S^2Q\Omega^{-1}Q \right]^{-1} \left[ X'Y + S^2Q\Omega^{-1}q \right]
\]

where,

\[
S^2 = \frac{1}{(n-p)} \{(Y - Xb_o)'(Y - Xb_o)\}
\]

is an unbiased estimator of \( \sigma^2 \).

If we drop the property of linearity and unbiasedness, we can obtain improved estimator of \( \beta \) in a number of ways. For instance, we may consider Stein – rule estimators.

\[
b_S = \left[ 1 - k \frac{(Y - Xb_o)'(Y - Xb_o)}{b_o'Cb_o} \right] b_o
\]

where \( k \) is the positive characterizing scalar.
Combining the idea of mixed estimation with that of Stein-rule, when there are some incomplete prior information, Srivastava and Srivastava (1983) proposed two families of improved estimators for $\beta$.

$$b_{sm} = \left[ C + \frac{1}{(n-p)} \{(Y - b_s)'(Y - b_s)\} Q' \Omega^{-1} Q \right]^{-1} \left[ X'Y + \frac{1}{(n-p)} \{(Y - b_s)'(Y - b_s)\} Q' \Omega^{-1} q \right]$$

$$b_{ms} = \left[ 1 - k \frac{(Y - Xb_m)'(Y - Xb_m)}{b_m'Cb_m} \right] b_m$$

where the estimator $b_{sm}$ is constructed by using stein-rule estimator in place of ordinary least squares in (2.5) and the estimator $b_{ms}$ is constructed by using mixed regression estimator in place of ordinary least squares estimator is (2.7).

3 Prediction within the Sample

Employing (2.4), (2.5), (2.8) and (2.9), we get the following four predictors for the values of the study variable within the sample.

$$\hat{T}_e = Xb_e$$

$$\hat{T}_m = Xb_M$$

$$\hat{T}_{sm} = Xb_{sm}$$

$$\hat{T}_{ms} = Xb_{ms}$$

In order to derive the properties of the predictors defined above, we shall employ small sample asymptotic theory introduced by Kadane (1971) and define the following notations:

$$P_\chi = X \Omega^{-1} X'$$

$$M = [I - P_\chi]$$

$$M_j = \left[ P_\chi - \frac{j}{\beta' \beta} X \beta \beta' X' \right] ; \quad j = 1, 2, \ldots$$

$$N_j = \left[ C^{-1} - \frac{j}{\beta' \beta} \beta \beta' \right] ; \quad j = 1, 2, \ldots$$

$$\mu = C^{-1} Q' \Omega^{-1} Q C^{-1}$$
It is easy to see that the predictor defined in (3.1) is unbiased with variance-covariance matrix and risk function respectively, defined as

\[(3.9) \quad \hat{V}(\hat{T}_o) = \sigma^2 \left\{ \alpha^2 I_n + (1 - 2\alpha) P_x \right\} \]

\[(3.10) \quad Risk(\hat{T}_o) = \sigma^2 \left\{ n\alpha^2 + (1 - 2\alpha) P \right\} \]

Since the properties of \( \hat{\tau}_u \) are identical to those of \( \hat{T}_{SM} \), so we consider the properties of \( \hat{T}_{SM} \) only.

**Theorem 1:** When disturbances are small and not necessarily normal, the bias vector, mean squared error matrix and risk function of the predictor \( \hat{T}_{SM} \) are given as

\[(3.11) \quad Bias(\hat{T}_{SM}) = - \sigma^3 \frac{\gamma_1}{(n-p)} X \mu X' (I_n \times M) e \]

where \(*\times*\) denotes Hadamard product and \(e\) is \((n\times1)\) vector with all elements unity.

\[(3.12) \quad MSE(\hat{T}_{SM}) = V(\hat{T}_o) - \sigma^4 \left\{ \frac{2\gamma_2}{(n-p)} \{tr \left( (P_x - \alpha) (I_n \times M) \mu C \right) - \frac{1}{2(n-p)} trM(I_n \times M) X \mu X' \} \right\} \]

\[(3.13) \quad Risk(\hat{T}_{SM}) = Risk(\hat{T}_o) - \sigma^4 \left\{ \frac{2\gamma_2}{(n-p)} \left\{ tr \left( (P_x - \alpha) (I_n \times M) \mu C \right) - \frac{1}{2(n-p)} trM(I_n \times M) X \mu X' \right\} \right\} \]

**Theorem 2:** When disturbances are small and not necessarily normal, the bias vector, mean squared error matrix and risk function of the predictor \( \hat{T}_{MS} \) are given as

\[(3.14) \quad Bias(\hat{T}_{MS}) = - \sigma^2 \frac{k(n-p)}{\beta \epsilon M} X \beta + \sigma^3 \frac{\gamma_1}{(n-p)} \{X \mu X' (I_n \times M) e + \frac{k(n-p)}{\beta \epsilon M} M \zeta \}

\( (I_n \times M) e \} + \sigma^4 \left\{ \frac{k}{\beta \epsilon M} \left\{ \gamma_2 \left\{ \frac{1}{\beta \epsilon M} \left( trM_4 (I_n \times M) I + 2P_x (I_n \times M) \right) X \beta - \frac{1}{(n-p)^2} trM(I_n \times M) X \mu C \} + \frac{(n-p)}{\beta \epsilon M} (p-2) X \beta + \frac{1}{(n-p)} X \beta \mu C \right\} \]

\[(3.15) \quad MSE(\hat{T}_{MS}) = V(\hat{T}_o) - \sigma^2 \frac{k \gamma_1 (1-\alpha)}{\beta \epsilon M} \left\{ \beta X' (I_n \times M) e + e' (I_n \times M) X \beta \right\} - \sigma^4 \left\{ \frac{\gamma_2}{(n-p)} \left\{ X \mu X' (I_n \times M) (P_x - \alpha) + (P_x - \alpha) (I_n \times M) X \mu X' - \frac{trM(I_n \times M)}{(n-p)} X \mu X' \right\} \right\} \]
\[ + \frac{k(n-p)}{\beta C\beta} \{ M_z (I_n^* M) (P_x - \alpha) + (P_x - \alpha) (I_n^* M) M_z - \frac{k}{\beta C\beta} \} \]

\[
\left( trM (I_n^* M) \right) X\beta\beta' X' \} + \{1-2(\alpha+\frac{1}{n-p})\} X\mu X' + \frac{k(n-p)}{\beta C\beta} .
\]

\[ 2(1-\alpha)M_z - \frac{k(n-p+2)}{(n-p)^2} X\beta\beta' X' \}
\]

(3.16) \[
Risk(\hat{T}_{MS}) = Risk(\hat{T}_o) - \sigma^2 \frac{2k\gamma(1-\alpha)}{\beta C\beta} \beta' X' (I_n^* M)e - \sigma^4 \left[ \frac{2\gamma_2}{(n-p)} \right] \left( tr(P_x - \alpha) \right).
\]

\[
(I_n^* M) \muC - \frac{trM(I_n^* M)}{2(n-p)} (tr\muC) + \frac{k(n-p)}{\beta C\beta} \{ tr ((P_x - \alpha) (I_n^* M) .
\]

\[
M_z - \frac{k}{2} \left( trM (I_n^* M) \right) \} + \{1-2(\alpha+\frac{1}{n-p})\} (tr\muC)
\]

\[
+ \frac{k(n-p)}{\beta C\beta} \left\{ 2(1-\alpha)(p-2) - \frac{k(n-p+2)}{(n-p)^2} \right\}
\]

4 A Comparison

On comparing the risks associated with the predictor \( \hat{T}_o \) and \( \hat{T}_{SM} \), we find that up to order of our approximation \( o(\sigma^4) \),

(4.1) \[
Risk(\hat{T}_o) - Risk(\hat{T}_{SM}) = \sigma^2 \left[ \frac{2\gamma_2}{(n-p)} \right] \left( \frac{trM ((P_x - \alpha) (I_n^* M) \muC}{2(n-p)} - \frac{1}{2(n-p)^2} \right).
\]

\[
\left( I_n^* M \right) X\mu X' \} + \{1-2(\alpha+\frac{1}{n-p})\} \muC \]

For normally distributed disturbances, we see that the predictor \( \hat{T}_{SM} \) is better than the predictor \( \hat{T}_o \) with respect to risk criterion, so long as

(4.2) \[
\alpha < \left( \frac{1}{2} - \frac{1}{n-p} \right)
\]

For actual values prediction (i.e., \( \alpha = 1 \)), the predictor \( \hat{T}_o \) is better than the predictor \( \hat{T}_{SM} \) and just reverse holds for average values prediction (i.e., \( \alpha = 0 \)).

On comparing the risks associated with the predictor \( \hat{T}_o \) and \( \hat{T}_{MS} \), we find that
(4.3) \[ \text{Risk}(\hat{T}_o) - \text{Risk}(\hat{T}_{ms}) = \sigma^3 \frac{2k\gamma_i(1-\alpha)}{\beta' CB} \beta' X' (I_n * M) e + \sigma^4 \left[ \frac{2\gamma_j}{(n-p)} \right] \left\{ (tr(P_x - \alpha) \right. \]
\[ \left. + \frac{trM(I_n * M)}{2(n-p)} (tr\mu C) + \frac{k(n-p)}{\beta'C\beta} \left\{ tr ((P_x - \alpha) (I_n * M)) \right. \]
\[ \left. + \frac{2(n-p)}{\beta'C\beta} \left\{ 2(1-\alpha)(p-2) - \frac{k(n-p+2)}{(n-p)^2} \right\} \right\} \]

up to the order of our approximation \( o(\sigma^4) \), we observe that for actual values prediction (i.e., \( \alpha = 1 \)), both the predictors have same risk and for average values prediction (i.e., \( \alpha = 0 \)), the predictor \( \hat{T}_{ms} \) has smaller risk than \( \hat{T}_o \) if, \( \gamma_i \) and \( \beta \) have the same sign otherwise reverse holds true. For normally distributed disturbances, we see that up to the order of our approximation \( o(\sigma^4) \), the predictor \( \hat{T}_{ms} \) dominates the predictor \( \hat{T}_o \) with respect to risk criterion, if \( k \) satisfies,

(4.4) \[ 0 < k < \frac{2(n-p)^2}{(n-p+2)} (1-\alpha)(p-2) ; \quad p > 2 \]

and expression (4.2) holds true. For actual values prediction (i.e., \( \alpha = 1 \)), \( \hat{T}_o \) have smaller risk than \( \hat{T}_{ms} \) and just reverse holds true for average values prediction (i.e., \( \alpha = 0 \)).

On comparing the risks associated with the predictor \( \hat{T}_{sm} \) and \( \hat{T}_{ms} \), we find that

(4.5) \[ \text{Risk}(\hat{T}_{sm}) - \text{Risk}(\hat{T}_{ms}) = \sigma^3 \frac{2k\gamma_i(1-\alpha)}{\beta' CB} \beta' X' (I_n * M) e + \sigma^4 \left[ \frac{2\gamma_j}{\beta'C\beta} \right] \left\{ tr ((P_x - \alpha) \right. \]
\[ \left. + \frac{trM(I_n * M)}{2(n-p)} (tr\mu C) + \frac{k(n-p)}{\beta'C\beta} \left\{ 2(1-\alpha)(p-2) \right. \]
\[ \left. - \frac{k(n-p+2)}{(n-p)^2} \right\} \right\]
actual values prediction (i.e., $\alpha = 1$), the predictor $\hat{T}_{sm}$ have smaller risk than the predictor $\hat{T}_{as}$ and reverse holds true for average values prediction (i.e., $\alpha = 0$).

5 Derivations of the Theorems

Using (2.4) in (2.1), we obtain the estimator

\begin{equation}
(5.1) \quad b_o = \beta + \sigma C^{-1} X' \omega
\end{equation}

\begin{equation}
(5.2) \quad Y - Xb_o = \omega M \omega
\end{equation}

So that up to order of our approximations $o(\sigma^2)$, we find

\begin{equation}
(5.3) \quad b_o' Cb_o = \beta' C \beta - 2 \sigma \beta' X' \omega + \sigma^2 \omega P_x \omega
\end{equation}

and for the same order of approximations,

\begin{equation}
(5.4) \quad [b_o' Cb_o]^{-1} = \frac{1}{\beta' C \beta} \left[ 1 - \frac{2 \sigma}{\beta' C \beta} \beta' X' \omega \right]
\end{equation}

using (5.1), (5.2), (5.4) in (2.7), we find that up to order of our approximation $o(\sigma^2)$,

\begin{equation}
(5.5) \quad b_3 = \beta + \sigma C^{-1} X' \omega - \sigma^2 \frac{k}{\beta' C \beta} \omega M \omega \beta
\end{equation}

and for the same order of approximations,

\begin{equation}
(5.6) \quad Y - Xb_3 = \omega M \omega + \sigma^2 \frac{k}{\beta' C \beta} \omega M \omega X \beta
\end{equation}

using (5.6) and (2.3) in (2.8), we obtain the Stein mixed estimator,

\begin{equation}
(5.7) \quad b_{sm} = \beta + \sigma C^{-1} X' \omega + \sigma^2 \frac{\omega' M \omega}{n - p} C^{-1} Q' \Omega^{-1} V - \sigma^3 \frac{\omega' M \omega}{n - p} \mu X' \omega - \sigma^4 (\omega' M \omega)^2.
\end{equation}

\begin{equation}
\left\{ \frac{1}{(n - p)^2} \mu - \frac{k^2}{\beta' C \beta} C^{-1} \right\} Q' \Omega^{-1} V
\end{equation}

using (5.7) in (2.2), we obtain the following expression up to the order of our approximation $o(\sigma^4)$,

\begin{equation}
(5.8) \quad \hat{T}_{sm} - T = \sigma \phi_1 + \sigma^2 \phi_2 + \sigma^3 \phi_3 + \sigma^4 \phi_4
\end{equation}

where

\begin{equation}
(5.9) \quad \phi_1 = (P_x - \alpha) \omega
\end{equation}
\( (5.10) \phi_2 = \frac{\omega M \omega}{n-p} X C^{-1} Q \Omega^{-1} V \)

\( (5.11) \phi_i = -\frac{\omega M \omega}{n-p} X \mu X^\prime \omega \)

\( (5.12) \phi_4 = - (\omega M \omega)^2 \frac{1}{(n-p)} X \mu - \frac{k^2}{\beta C \beta} X C^{-1} Q \Omega^{-1} V \)

Here, it is easy to see that

\( (5.13) \) \( E[\phi_i] = E[\phi_1] = E[\phi_4] = 0 \)

\( (5.14) \) \( E[\phi_1] = \frac{-\gamma_i}{(n-p)} X \mu X^\prime (I_n \ast M)e \)

Utilizing (5.13) and (5.14) in (5.8), we obtain the expression (3.11) of Theorem 1.

Now

\( (5.15) \) \( [\hat{T}_{SM} - T] [\hat{T}_{SM} - T]^\prime = \sigma^2 \phi_1 \phi_1^\prime + \sigma^4 (\phi_2 \phi_2^\prime + \phi_3 \phi_3^\prime) + \sigma^4 (\phi_4 \phi_4^\prime + \phi_5 \phi_5^\prime + \phi_6 \phi_6^\prime) \)

Here,

\( (5.16) \) \( E[\phi_1 \phi_1^\prime] = \{ \alpha^2 I_n + (1-2\alpha)P_s \} \)

\( (5.17) \) \( E[\phi_2 \phi_2^\prime] = 0 \)

\( (5.18) \) \( E[\phi_3 \phi_3^\prime] = \frac{-\gamma_2}{(n-p)} \{ (P_x - \alpha) (I_n \ast M) X \mu X^\prime + (n-p)(1-\alpha)X \mu X^\prime \} \)

\( (5.19) \) \( E[\phi_4 \phi_4^\prime] = \frac{1}{(n-p)} \{ \gamma_2 trM (I_n \ast M) + (n-p)(n-p+2) \} X \mu X^\prime \)

Utilizing (5.16), (5.17), (5.18) and (5.19) in (5.15), we obtain the expression (3.12) of Theorem 1.

Again

\( (5.20) \) \( \text{Risk}(\hat{T}_{SM}) = E[\hat{T}_{SM} - T] [\hat{T}_{SM} - T]^\prime = E[tr[\hat{T}_{SM} - T] [\hat{T}_{SM} - T]^\prime] = trMSE(\hat{T}_{SM}) \)

And thus expression (3.13) of Theorem 1 is obtained by utilizing (5.20).

Using (5.2), (2.3) in (2.5), we obtain the mixed regression estimator up to of our approximation \( o(\sigma^2) \)

\( (5.21) \) \( b_M = \beta + \sigma C^{-1} X^\prime \omega + \sigma^2 \frac{\omega M \omega}{n-p} C^{-1} Q \Omega^{-1} V \)
(5.22) \( Y - Xb_m = \sigma M \omega - \sigma^2 \frac{\omega M \Omega}{n-p} \Omega^{-1} V \)

Employing (5.21), we obtain the following expression up to the our approximation \( o(\sigma^4) \),

(5.23) \( b_m' Cb_m = \beta' C \beta \left[ 1 + \frac{2\sigma}{\beta' C \beta} \beta' X' \omega + \sigma^2 \frac{1}{\beta' C \beta} \omega' \frac{2}{n-p} \omega M \omega' \Omega^{-1} Q \beta + \omega' M \omega \right] \)

(5.24) \[ b_m' Cb_m \] \( = \frac{1}{\beta' C \beta} \left[ 1 - \frac{2\sigma}{\beta' C \beta} \beta' X' \omega - \sigma^2 \frac{1}{\beta' C \beta} \omega' \frac{2}{n-p} \omega M \omega' \Omega^{-1} Q \beta + \omega' M \omega \right] \)

Using (5.21), (5.22) and (5.24) in (2.9), we obtain the mixed stein rule estimator up to order of our approximation \( o(\sigma^4) \)

(5.25) \( b_m = \beta + \sigma C^{-1} X' \omega + \sigma^2 \frac{\omega M \omega}{n-p} \left[ \frac{k(n-p)}{\beta' C \beta} - \sigma^3 \frac{\omega M \omega}{n-p} \right] \mu + \frac{k(n-p)}{\beta' C \beta} \left[ \frac{k}{\beta' C \beta} \omega' \frac{2}{n-p} \beta' Q \Omega^{-1} V + \omega' M \omega \beta \right] \)

\[ + 2C^{-1} X' \omega' M \omega X \beta \] \( - \frac{\omega M \omega}{(n-p)} \left[ C^{-1} Q \Omega^{-1} Q + \frac{k}{\beta' C \beta} \left( \beta' V' \Omega^{-1} Q + (n-p) I \right) \right]. \]

Using (5.25) in (2.2), we obtain the following expression up to the same order of approximation,

(5.26) \( \hat{T}_{ms} - T = \sigma \eta_1 + \sigma^2 \eta_2 + \sigma^3 \eta_3 + \sigma^4 \eta_4 \)

where

(5.27) \( \eta_1 = (P_x - \alpha) \omega \)

(5.28) \( \eta_2 = \frac{\omega M \omega}{n-p} \left[ XC^{-1} Q \Omega^{-1} V - \frac{k(n-p)}{\beta' C \beta} X \beta \right] \)

(5.29) \( \eta_3 = -\frac{\omega M \omega}{n-p} \left[ \frac{k}{\beta' C \beta} X \mu X + \frac{k(n-p)}{\beta' C \beta} M \omega \right] \)

(5.30) \( \eta_4 = \omega M \omega \left[ \frac{k}{\beta' C \beta} \left( \frac{2 \omega M \omega}{n-p} X \beta \beta' Q \Omega^{-1} V + \omega' M \omega' X + 2P_x \omega \omega' X \beta \right) \right] - \frac{\omega M \omega}{(n-p)} \left[ XC^{-1} Q \Omega^{-1} Q + \frac{k}{\beta' C \beta} \left( \beta' V' \Omega^{-1} Q + (n-p) I \right) \right] \)

Here,

(5.31) \( E[\eta] = 0 \)
(5.32) $E[\eta] = -\frac{k(n-p)}{\beta' C\beta} X\beta$

(5.33) $E[\eta] = -\frac{1}{(n-p)}\left\{X\mu X' (I_n * M)e + \frac{k(n-p)}{\beta' C\beta} M (I_n * M)e\right\}$

(5.34) $E[\eta] = \frac{k}{\beta' C\beta} \left[ \gamma_2 \left\{ \frac{1}{\beta' C\beta} \left( trM + (I_n * M)I + 2P_x (I_n * M) \right) X\beta \right. \\
- \frac{trM (I_n * M)}{(n-p)^2} X\beta \mu C \right] + \frac{(n-p)}{(n-p)(p-2)} X\beta \left( \frac{n-p+2}{n-p} \right) X\beta \mu C \right]$

Utilizing (5.31), (5.32), (5.33) and (5.34) in (5.26), we obtain the expression (3.14) of Theorem 2.

Now,

(5.35) $[\hat{T}_{\text{MS}} - T][\hat{T}_{\text{MS}} - T]' = \sigma^2 \eta_1 \eta_1' + \sigma^3 (\eta_1 \eta_2' + \eta_2 \eta_1') + \sigma^4 (\eta_1 \eta_3' + \eta_2 \eta_2' + \eta_3 \eta_1')$

Here,

(5.36) $E[\eta \eta_1'] = \{ \alpha^2 I_n + (1-2\alpha)P_x \}$

(5.37) $E[\eta \eta_2'] = -\frac{k\gamma_1(1-\alpha)}{\beta' C\beta} \beta' X' (I_n * M)e$

(5.38) $E[\eta \eta_3'] = -\left[ \frac{\gamma_2}{(n-p)} \left\{ (P_x - \alpha) (I_n * M) X\mu X' + \frac{k(n-p)}{\beta' C\beta} (P_x - \alpha) (I_n * M) M \right\} \\
+ \frac{(n-p)(1-\alpha)}{\beta' C\beta} \left\{ \frac{1}{\beta' C\beta} \right\} \left( trM + (I_n * M) \right) X\beta' X' \left\{ \frac{k^2}{\beta' C\beta} \right\} \right]$

(5.39) $E[\eta \eta_2'] = \gamma_2 \left\{ \frac{trM (I_n * M)}{(n-p)^2} X\mu X' + \frac{k^2}{\beta' C\beta} \left\{ trM (I_n * M) \right\} X\beta' X' \right\}$

Utilizing (5.36), (5.37), (5.38) and (5.39) in (5.35), we obtain the expression (3.15) of Theorem 2.

Similarly,

(5.40) $\text{Risk}(\hat{T}_{\text{MS}}) = E [ \hat{T}_{\text{MS}} - T ] [ \hat{T}_{\text{MS}} - T ] = \text{trMSE}(\hat{T}_{\text{MS}})$

Thus expression (3.16) of Theorem 2 is obtained by utilizing (5.40).
References


Toutenburg, H. and G. Trenkler (1990), "Mean square error matrix comparison of optimal and classical predictors and estimators in linear regression", Computational Statistics and Data Analysis, 10, 297 – 305.