

## Introduction

It is uncommon for Dover Publications to reprint a conference proceedings. However, there is nothing common about *Topology of 3-Manifolds and Related Topics*. Its forty-seven papers communicate the ideas as well as the spirit of a significant era of topology. Survey articles by legendary mathematicians such as R.H. Bing, Morton Brown, S.S. Cairnes and E.C. Zeeman carefully review ideas from decomposition theory and manifold structures. The most notable inclusion is Ralph Fox's "Quick Trip Through Knot Theory," reprinted here for the first time. It is a masterly and beloved survey that remains one of the best introductions to the study of knots and links.

Other, more specialized articles in this carefully edited volume range from concise summaries of talks to extensive research articles. Students and seasoned mathematicians alike will discover gems in this unusual volume. "On Fiberings of Certain 3-Manifolds," by John Stallings, is a stone that shines especially bright. Its main result, known today as the Stallings fibration theorem, is a fundamental tool for understanding the structure of many 3-manifolds, one that is cited throughout the literature.

Readers with an interest in the applications of topology will appreciate "The Topology of the Brain and Visual Perception," one of five contributions by Zeeman. In it, he proposes isomorphic models for the process of thinking and for brain function. Interest in this topic and mathematical models for the brain generally is accelerating today in the field of cognitive psychology.

By the summer of 1961, geometric topology had enjoyed its first sunrise of a new golden age, one that we still enjoy today. During the previous year, Stephen Smale had announced a proof of the generalized Poincaré conjecture: a piecewise-linear or differentiable  $n$ -dimensional manifold that is homotopy-equivalent to the sphere  $S^n$  is homeomorphic to  $S^n$ , provided that  $n > 4$ .

Smale's argument as well as those of Andrew Wallace, Stallings and Zeeman, which came quickly afterward, demonstrated a general principle that seemed almost unbelievable. Manifolds of high dimension are easier to understand than those of dimensions 3 and 4. Part of the explanation for this is found in the fundamental group. Regardless of the dimension of the manifold, relations among generators can be represented by 2-dimensional disks. When the dimension is greater than 4, the disks are generically embedded and disjoint from each other. Using the "Whitney trick," not a magic act but rather a clever geometric device named for its inventor, Hassler Whitney, structures can be simplified, and theorems proved.

Manifolds of dimension 3, called simply "3-manifolds," have a natural appeal. Most of us believe that we live in one, but exactly which one remains a deep mystery. During the decade preceding the 1961 Georgia Topology Institute, Edwin Moise had shown that poorly-understood distinctions among topological, piecewise-linear and differentiable structures on manifolds disappear in this low dimension.

However, 3-manifolds present other, special difficulties that are unique to that dimension. While their fundamental groups are more tightly related to their topologies than is the case in higher dimensions, working with those groups can be frustratingly difficult. Just a few years before the summer of 1961, encouragement came. Chirstos Papakyriakopoulos (affectionately called "Papa"), a Greek topologist who had been brought to Princeton University ten years earlier by Fox, succeeded in establishing a foundational result that had been claimed but not proved many years earlier by Max Dehn: if a simple closed curve on the boundary of a 3-manifold bounds a singular disk in the manifold, then it bounds an embedded disk. His proof employed an ingenious technique, a "tower construction," that enabled him to fashion two more important tools, the loop theorem and the sphere theorem. The study of surfaces in 3-manifolds would advance immeasurably.

The 1961 Georgia Topology Institute was preceded by an 8-week Topology Institute for Graduate Students, mostly from the southern United States. Lecture duties were shared between Bing and Deane Montgomery. Montgomery, who lectured on topological groups, Lie groups and group actions, had helped solve Hilbert's fifth problem only a few years before. Bing distributed lecture notes as well as problems to be solved, and he encouraged the students to make their own conjectures and try to prove them. Photocopied notes were exotic at that time, and Beverly Brechner, now Professor Emeritus at University of Florida, remembers how exciting it was to receive them. "After these two months, I was hooked," she recalls. "Nothing could make me quit mathematics!" Although Montgomery had to leave at the end of the graduate student workshop, Bing was able to remain for the August conference. Eight of the 24 participating graduate students, including Brechner, were invited to stay.

Smale's proof of the generalized Poincaré conjecture was only one of several developments that motivated the 1961 Georgia Topology Institute. During the previous year, Zeeman had proved that any combinatorial

sphere  $\mathbb{S}^n$  in Euclidean space bounds a simplex if the dimension of the ambient space exceeds  $\frac{3}{2}(n+1)$ . This result, known today as “Zeeman’s unknotting theorem,” established as a corollary that knot theory is trivial in codimensions greater than 2. Research would now focus on  $n$ -spheres in space of dimension  $n+2$ , with 1-spheres (or “knots”) in 3-dimensional space being the interesting case of lowest dimension.

Zeeman recalls two other breakthroughs that stirred participants at the 1961 conference. In 1956, John Milnor had startled the topological community by proving that the 7-dimensional sphere supports more than one differentiable structure. A few years later, in 1960, Brown gave an elegant new proof of the generalized Schoenflies theorem: a locally flat  $(n-1)$ -sphere  $\Sigma$  in the  $n$ -sphere  $\mathbb{S}^n$  is standard in the sense that it can be carried to the equator by a homeomorphism. (The differentiable version had been shown just one year before by Barry Mazur.) The hypothesis that  $\Sigma$  be locally flat means that for each point of  $\Sigma$  there is a neighborhood  $N$  and a homeomorphism of  $N$  into  $\mathbb{S}^n$  such that  $N \cap \Sigma$  is contained in the equator  $\mathbb{S}^{n-1}$ . Such a hypothesis was necessary in view of J. Alexander’s “horned sphere” of 1924, one of the most famous pathological examples in topology. It is a non-locally flat 2-sphere in  $\mathbb{S}^3$  that bounds a ball on one side but not the other; in particular, it cannot be carried to the equator by a homeomorphism. Zeeman recalls that Alexander’s example discouraged those working in low-dimensional topology. What was especially significant about Brown’s argument was that it did not require triangulations. In Zeeman’s words, “It established the true topology.”

Smale’s proof of the generalized Poincaré conjecture, Zeeman’s unknotting theorem, Milnor’s construction of exotic 7-spheres and Brown’s proof of the Schoenflies theorem did more than encourage research in geometric topology. Each was responsible for a new school of thought. Algebraic topology, which, in Zeeman’s words, had “eaten up topology,” would now give room to the newly strengthened field of geometric topology with its pioneering spirit.

In the years that followed the 1961 Georgia Topology Institute, deep discoveries would revitalize geometric topology and take it in unexpected directions. In 1982, William Thurston would conjecture that a compact 3-manifold can be decomposed into submanifolds, each having one of eight geometric structures. Thurston would prove his “geometrization conjecture” for Haken 3-manifolds, a large and important class of 3-manifolds. The general conjecture, which implies the Poincaré conjecture, that any 3-manifold homotopy-equivalent to the 3-sphere is homeomorphic to it, would be proved in 2003 by Grigori Perelman, using Richard Hamilton’s notion of Ricci flow. In the meantime, Thurston’s ideas would have profound implications for geometric topology, especially knot theory.

The close relationship between knots and 3-manifolds had been known for at least a century. However, it was just one year before the 1961 Georgia Topology Institute that Wallace proved that any closed oriented 3-manifold can be obtained from the 3-sphere by removing a finite collection of mutually disjoint solid toral neighborhoods of knots and sewing them back differently. (Max Dehn used this idea in 1910 to produce manifolds with the same homology as the 3-sphere.) The reader will find John Hempel’s argument here. It has the added feature that all of the solid tori that are removed can be taken to be unknotted.

It is understandable that knot theory became enormously popular by the time of the 1961 Georgia Topology Institute. Ralph Fox was greatly responsible. The room in which he ran his seminar at Princeton University never seemed to have to have enough chairs, according to Wilbur Whitten, who received his Ph.D. under Fox in 1961. Other doctoral students of Fox include Herman Gluck, Barry Mazur, Milnor, Lee Neuwirth and Stallings.

Fox’s classic survey, “A Quick Trip Through Knot Theory,” expertly surveys much of the subject at the time. Knot groups, the free differential calculus, and Alexander polynomials are presented. A brief primer of covering spaces is found here. Seifert surfaces are used to construct cyclic and branched coverings of knots. Representations of knot groups into finite groups, particularly metacyclic groups, are constructed for the first time. The writing is fresh and clear. Such a survey was destined to become a perennial favorite.

Every knot bounds an embedded disk in the closed upper half-space of 4 dimensions. Such a disk is simply a cone on the knot. Some knots, however, bound locally flat disks. In “A Quick Trip Through Knot Theory,” Fox displayed the first examples, which he called “slice knots.” (The name was suggested to Fox by Moise.) Fox and Milnor had just recently shown that any slice knot has an Alexander polynomial of the form  $f(t)f(t^{-1})$ , where  $f(t)$  is a polynomial with integer coefficients. Consequently, not every knot is a slice knot. The idea brought the study of knots into close contact with that of 4-dimensional manifolds.

Knot theory has undergone many changes since the 1961 Georgia Topology Institute. Perhaps the most

profound change was brought about in 1983 by Vaughan Jones, who discovered a powerful new knot invariant from the perspective of operator algebras. The subsequent combinatorial path that the subject took opened up new connections with physics, still not well understood. But in recent years the classical ideas of knot theory have merged with the new, combinatorial methods. Students will benefit from reading Fox's survey paper, if only by gaining a classical perspective of this fast-developing subject.

The 1961 Georgia Topology Institute hoped to generate questions and conjectures for future research, and it did so with great zeal. In the years since then, many of them have been settled, the Poincaré conjecture being the most notable. Others remain open. For example, Fox observed that all of the slice knots that he had displayed in fact are boundaries of nicely immersed ribbons in 3-dimensional space. He asked if this is always the case. Fox's question, "Is every slice knot a ribbon knot?" has repelled all assaults so far.

One of the universal lessons of mathematics is that progress on even the most difficult problems is achievable. The participants of the 1961 Georgia Topology Institute shared a feeling of optimism about the study of 3-manifolds, a feeling that has been justified repeatedly during the decades that followed. Sharing not just its knowledge but also its optimism is the aim of this book.