Introduction to the Dover Edition

"A frog is at the bottom of a 30-foot well." Is it the beginning of a joke? There might be a princess down there too. But the story continues: "Each hour he climbs 3 feet and slips back 2. How many hours does it take to get out?"

A mathematical riddle is much like a joke. Finding the answer requires us to look at things from more than one perspective. (It does not take the frog 30 hours to get out. Can you find the correct answer?) And as with a joke, our sudden understanding might cause us to laugh.

The book before you is a treasury of mathematical riddles. Some are classics. ("Brothers and sisters have I none, but that man's father is my father's son." What is the relation of "that man" to the speaker?) Others might be less familiar. ("Two points are chosen at random on the surface of a sphere. What is the probability that the distance between them is less than 10 minutes of an arc?") But, like a great joke, a classic riddle is always a pleasure to hear when told by a master.

The subtitle of *Riddles in Mathematics* is *A Book of Paradoxes*. What is a paradox? The author, Eugene Northrop, anticipates our question and responds in the first few pages. A paradox is "anything which offhand appears to be false but is actually true; or which appears to be true, but is actually false; or which is simply self-contradictory." He writes: "From time to time it may appear that we are straying from this meaning. But be patient—what seems crystal-clear to you may leave the next person completely confused." The ability to make things crystal-clear was the author's talent.

Like the author, most mathematicians appreciate good paradoxes. Without the temporary confusion that they create, no new understanding is possible. Put differently, a paradox helps us correct our thinking. It flushes from our minds flawed intuition and hazardous mental debris. Physicist Richard Feynmann expressed it another way: "A paradox is not a conflict within reality. It is a conflict between reality and your feeling of what reality should be like."

Paradoxes are not new in mathematics. In the seventeenth century, Galileo Galilei pondered his discovery that squared integers 1, 4, 9, 16, . . . can be put into one-to-one correspondence with the larger set of natural numbers 1, 2, 3, 4, . . . which contains them. It was not the only paradox of the infinite that he observed and wrote about in *Dialogues Concerning the Two New Sciences.* By imagining the hub and rim of a wheel shrinking in area simultaneously, Galileo came to the disturbing conclusion that *a single point is equal to the circumference of a circle.* Northrop sorts out Galileo's confusion (alas, three centuries too late to help the Renaissance scholar) and then proceeds to baffle us with other paradoxes of infinity.

The author of *Riddles in Mathematics* was born in Danbury, Connecticut in 1908. For two years he attended Robert College in Istanbul, Turkey, where he decided to switch his major from civil engineering to mathematics. He returned to the United States and completed his undergraduate, master's and Ph.D. degrees at Yale University. Northrop was the first student of the distinguished analyst Einar Hille. Much of Hille's energies at Yale were devoted to improving undergraduate education, and he must have inspired his student to follow his example. Throughout his career, Northrop regarded himself primarily as a mathematics educator. After Yale, he taught at Hotchkiss School, a private boarding school in Connecticut. Most of *Riddles in Mathematics* was written while Northrop was there. The book was published shortly after he left and began teaching at University of Chicago.

Riddles in Mathematics appeared in March, 1944. The United States, which had been at war for more than three years, faced new, urgent demands. Soldiers needed to perform mathematical tasks such as map reading, setting fuses and computing airspeeds. Even army cooks, who had to order supplies and prepare a staggering number of meals, needed mathematics. The military had discovered that many of its recruits could barely do arithmetic.

A powerful nation can build battleships and planes in a hurry, but it cannot easily correct for decades of classroom neglect. Northrop and his colleagues at University of Chicago worked to reshape curricula in ways that would influence both colleges and high schools in the decades to come. Northrop would serve as a consultant to the fledgling National Science Foundation for more than a year, helping to develop programs for retraining teachers in mathematics and science.

The years leading up to the war had blown out the boundaries of mathematics. Logic, probability and topology had all moved in surprising directions. *Riddles in Mathematics* is a time capsule from one of the most stimulating periods in the history of mathematics, now reopened after 70 years.

"Logic has become more mathematical and mathematics has become more logical," wrote Bertrand Russell in 1919. "The consequence is that it has now become wholly impossible to draw a line between the two; in fact, the two are one." The chapter on logic begins: "All statements made by Cretans are false." Nothing seems paradoxical about it until you learn that it was issued by a Cretan, Epimenides, a poet and prophet who lived 26 centuries ago. If Epimenides was telling the truth, then he was lying. If he was lying, then he was telling the truth. Northrop follows the progeny of this famous paradox, arriving finally at Russell's "theory of logical types."

General topology also spawned paradoxes of a kind that had not previously been seen. We hear about "bottles" that can hold no liquid and continuous paths that visit every point in space. Glue together the ends of a long paper strip that has been given a half-twist, and the result is called a Möbius strip. Cut it down the middle and get a single long loop twice as long as the original. For some, that is paradoxical enough. However, cutting a strip that has had three half-twists placed in it yields a surprise. (If you don't know what the surprise is, go get paper, scissors and tape. If you are accident-prone, then just read pages 72–73.)

Like any classic, *Riddles in Mathematics* remains relevant and fresh today. Nevertheless, a few of its assertions can be updated. One of them concerns prime numbers. A *prime number* (or simply a *prime*) is a positive integer greater than 1 that has no positive divisors other than 1 and itself. Euclid provided a short, elegant proof of the infinitude of primes. Later, Eratosthenese of Cyrene showed how all primes up to a given limit could be found. Still, the search for very large primes remains a challenge. In 1944, the largest prime number known was $2^{227} - 1$, a number with merely 39 digits. Thanks to the Great Internet Mersenne Prime Search, which began in 1996, we know much larger primes today. The current record-holder is $2^{57885161} - 1$, a number with more than 17 million digits!

Mathematicians have made progress on another challenging problem in *Riddles of Mathematics*, the problem of coloring maps with few colors. It originated in an English classroom more than 150 years ago: "A student of mine asked me today to give him a reason for a fact which I did not know was a fact—and do not yet." So began a letter of 1852 to the Irish mathematician William Rowen Hamilton. The writer was Augustus de Morgan, a British mathematician and, like Northrop, a passionate collector of paradoxes. The "fact" had to do with maps, real or imaginary. It was obvious that some maps required at least four colors if no two neighboring regions would be colored the same. But for what reason are four colors always sufficient? De Morgan believed that four colors suffice, but belief and proof are not the same thing. No proof was known in 1944, when Northrop wrote about the so-called "four-color problem." It would come three decades later, in 1976. Using a computer, Kenneth Appel and Wolfgang Haken gave a proof that four colors do indeed suffice to color any map.

One of the many attractive features of *Riddles in Mathematics* is that it speaks to everyone. Students and teachers will find enough puzzles and false proofs to drive each other crazy. Searching for the flaw in Northrop's "proofs" that $\sqrt{a} + \sqrt{b} = \sqrt{2(a + b)}$ or that $45^\circ = 60^\circ$ are true exercises in critical thinking. What better way can there to sharpen algebra or trigonometry skills?

Northrop originally chose a more provocative title: *Two and Two Make Five*. However, the sales department at D. Van Nostrand Company, the first publishers of the book, were terrified by the prospect of promoting anything with such a moniker. They suggested a dreadful alternative, *Tricks with Figures*, thereby inflicting upon the author an equal share of terror. A long ping-pong match with names followed. In the end, Northrop served up the winner, *Riddles in Mathematics – A Book of Paradoxes*, a most appropriate title.

Which reminds me of a joke. An anthropologist exploring a jungle encountered a group of indigenous children being taught to tie knots in rope. When she heard that they were learning arithmetic, she begged their teacher to explain the method. "It's easy," the teacher replied. "For example, if I want to compute two plus two, then I tie two knots in this piece of rope and then two knots in a second piece of rope. I tie the two pieces together. Now I count all the knots: Two plus two make five!"

No matter what title one prefers, *Riddles in Mathematics* represents the best in popular mathematics writing.

DANIEL S. SILVER

University of South Alabama
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Preface

Popular interest in mathematics is unquestionably increasing. Perhaps this is because of the fact that mathematics is a tool without which the applied sciences would cease to be sciences. On the other hand, the abstract aspect of mathematics is beginning to attract a large following of people who, weary of the complexities of the human equation in everyday activities, turn in their leisure to the simplicities of the mathematical equation. It is for these people that this book is written. Indeed, only two things are required of the prospective reader—an elementary training in mathematics, and an interest in matters mathematical. These two prerequisites are sufficient for an understanding of the first nine chapters of the book. The tenth—and last—chapter is specifically designed for the reader with more technical equipment.

Of all the problems dealt with in mathematics, paradoxes are among the most appealing and instructive. The appeal of a paradox is difficult to analyze in a word or two, but it probably arises from the fact that a contradiction comes as a complete surprise in what is generally thought of as the only "exact" science. And a paradox is always instructive, for to unravel the troublesome line of reasoning requires a close scrutiny of the fundamental principles involved. In the light of these arguments it has seemed worth while to bring out a book devoted exclusively to some of the paradoxes which mathematicians, both amateur and professional, have found disconcerting.

The material for this book has been gathered from a wide variety of sources. Some of it has naturally appeared in other popular expositions of mathematics—such works as Ball's *Mathematical Recreations and Essays*, Steinhaus' *Mathematical Snapshots*, and Kasner and Newman's *Mathematics and the Imagination*, to mention only three. If this is a fault, it is not the fault of the author, but of the material he is try-