Vertex-Colored Graphs, Bicycle Spaces and Mahler Measure

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August 21, 2014

Abstract

The space $\mathcal{C}$ of conservative vertex colorings (over a field $F$) of a countably locally-finite graph $G$ is introduced. The subspace $\mathcal{C}^0$ of based colorings is shown to be isomorphic to the bicycle space of the graph. For graphs $G$ with a free $\mathbb{Z}^d$-action by automorphisms, $\mathcal{C}$ is a finitely generated module over the polynomial ring $F[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$, and for this a polynomial invariant, the Laplacian polynomial, $\Delta_0$ is defined. Properties of $\Delta_0$ are discussed. The logarithmic Mahler measure of $\Delta_0$ is characterized in terms of the growth of spanning trees of $G$.

1 Introduction

Graphs have been an important part of knot theory investigations since the nineteenth century. In particular, finite plane graphs correspond to alternating links via the medial construction (see section 4). The correspondence became especially fruitful in the mid 1980’s when the Jones polynomial renewed the interest of many knot theorists in combinatorial methods while at the same time drawing the attention of mathematical physicists.

Coloring methods for graphs also have a long history, one that stretches back at least as far as Francis Guthrie’s Four color conjecture of 1852. By contrast, coloring techniques in knot theory are relatively recent, mostly motivated by an observation of in the 1963 textbook, *Introduction to knot theory*, by Crowell and Fox. In view of the relationship between finite plane graphs and alternating links, it is not surprising that a corresponding theory of graph coloring exists. This is our starting point. However, by allowing nonplanar graphs and also countably infinite graphs that are locally finite, a richer theory emerges.

Section 2 introduces the space $\mathcal{C}$ of conservative vertex colorings (over a field $F$) of a countably locally-finite graph $G$. We identify the subspace $\mathcal{C}^0$ of based conservative vertex colorings with the bicycle space $\mathcal{B}$ of $G$. In section 3 we define the space of conservative edge colorings of $G$, which we show is naturally isomorphic to $\mathcal{C}^0$. When $G$ is embedded in the plane, yet a third type of coloring, coloring vertices and faces of $G$, is possible. The resulting space, called the Dehn colorings of $G$, is shown to be isomorphic to the space $\mathcal{C}$. We use it to extend and sharpen the known result that residues of the medial link components generate $\mathcal{B}$.

When $G$ admits a free $\mathbb{Z}^d$-action by automorphisms, the vertex coloring space becomes a finitely generated module over the polynomial ring $F[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. Techniques of commutative algebra are used to define the Laplacian polynomial $\Delta_0$ of $G$. In sections 5 and 6 we consider infinite graphs that admit free $\mathbb{Z}$- and $\mathbb{Z}^2$-actions by automorphisms. One may think of such a graph as the lift to the universal cover of a finite graph $\bar{G}$ in the annulus or torus. When $G$ is a locally finite plane graph with free $\mathbb{Z}$-symmetry and $F$ is the 2-element field, we prove that the degree of the Laplacian

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*The second and third authors are partially supported by the Simons Foundation.*
polynomial is twice the number of noncompact components of the medial graph of $G$; when $\mathbb{F} = \mathbb{Q}$, the degree is shown to be the minimum number of vertices that must be deleted from $\bar{G}$ in order to enclose the graph in a disk.

In section 7.3 we enter the realm of algebraic dynamical systems. Using a theorem of D. Lind, K. Schmidt and T. Ward [20], [25], we characterize the logarithmic Mahler measure of the Laplacian polynomial in terms of the growth of the number of spanning trees of $G$. We obtain results about thermodynamic limits (also called bulk limits) that have previously been computed by purely analytical methods involving partition functions. The approach here is relatively simple.

We are grateful to Oliver Dasbach, Iain Moffatt and Lorenzo Traldi for their comments and suggestions.

2 Conservative vertex colorings

Throughout, $G$ is assumed to be a countable, connected, locally finite graph. We denote the field of $p$ elements by $GF(p)$.

Definition 2.1. A vertex coloring of $G$ is an assignment of elements (called colors) of a field $\mathbb{F}$ to the vertices of $G$. A vertex coloring is conservative if, for every vertex $v \in V$, the Laplacian vertex condition holds:

$$d \cdot \alpha = \sum_{i=1}^{d} \alpha_i,$$  \hspace{1cm} (2.1)

where $d$ is the degree of $v$, $\alpha$ is the color assigned to $v$, and $\alpha_1, \ldots, \alpha_d$ are the colors assigned to vertices adjacent to $v$, counted with multiplicity in case of multiple edges. (A self-loop at $v$ is counted as two edges from $v$ to $v$.)

The set of vertex colorings will be identified with $\mathbb{F}^V$, where $V$ is the vertex set of $G$. Conservative vertex colorings of $G$ form a vector space under coordinate-wise addition and scalar multiplication. We denote the vector space by $C$.

We will say two vertex colorings are equivalent if they differ by a constant vertex coloring. The constant vertex colorings are a subspace of $C$. We denote the quotient space by $C^0$, and refer to it as the space of based vertex colorings of $G$. If we select a base vertex of $G$, then an element of $C^0$ is represented by a unique vertex coloring that assigns 0 to that vertex.

Assume that the vertex and edge sets of $G$ are $V = \{v_1, v_2, \ldots\}$ and $E = \{e_1, e_2, \ldots\}$, respectively. The Laplacian matrix $L = (L_{ij})$ is $D - A$, where $D$ is the diagonal matrix of vertex degrees, and $A$ is the adjacency matrix.

If $V$ is infinite, then $L$ is a countably infinite matrix. Nevertheless, each row and column of $L$ has only finitely many nonzero terms, and hence many notions from linear algebra remain well defined. In particular, we can regard $L$ as an endomorphism of the space $\mathbb{F}^V$ of vertex colorings of $G$. The following is immediate.

Proposition 2.2. The space $C$ of conservative vertex colorings of $G$ is the kernel of the Laplacian $L$.

We orient the edges of $G$. The incidence matrix $Q = (Q_{ij})$ is defined by

$$Q_{ij} = \begin{cases} 1 & \text{if } e_i \text{ terminates at } v_i, \\ -1 & \text{if } e_j \text{ originates at } v_i, \\ 0 & \text{else} \end{cases}$$  \hspace{1cm} (2.2)
One checks that $L = QQ^T$, regardless of the choice of orientation.

**Definition 2.3.** The cut space or cocycle space $W$ of $G$ (over $F$) is the subspace of $F^E$ consisting of all, possibly infinite, linear combinations of row vectors of $Q$.

We regard $Q$ as a linear mapping from $F^V$ to $F^E$. Its kernel is 1-dimensional, spanned by the vector with constant coordinates equal to 1 (see, for example, Theorem 8.2.1 of [10]).

Denote by $\cdot$ the standard inner product on $F^E$; that is, $u \cdot w = \sum u_iw_j$, whenever only finitely many summands are nonzero.

**Definition 2.4.** The cycle space of $G$ (over $F$) is the space of all $u \in F^E$ with $w \cdot u = 0$ for every $w \in W$ for which the product is defined. We denote this space by $W^\perp$. The bicycle space $B$ is $W \cap W^\perp$.

A vector $w \in F^E$ is in the cut-space $W$ if and only if it is in the image of $Q$. Moreover, $w \in W^\perp$ if and only if $wQ^T = 0$. Since the kernel of $W$ consists of the constant vertex colorings of $G$, we have:

**Proposition 2.5.** The based vertex coloring space $C^0$ is isomorphic to the bicycle space $B$.

**Remark 2.6.** When $F = GF(2)$, vectors $w \in F^E$ correspond to subsets of $E$: an edge is included if and only if its coordinate in $w$ is nonzero. Vectors $w$ such that $wQ^T = 0$ correspond to subgraphs in which every vertex has even degree. Bonnington and Richter [1] call any such subgraph a cycle of $G$, which explains the term “cycle space.” The term “bicycle” arises since cuts in $B$ are also cocycles.

The reader is warned that there is not complete agreement in the literature about the definition of “cycle” for infinite graphs. (See for example [24].)

### 3 Conservative edge colorings

An element of $F^E$ may be regarded as a coloring of the edge set. Bicycles are edge colorings that satisfy two conservation laws.

**Definition 3.1.** An edge coloring of $G$ is an assignment of colors to the oriented edges of the graph. An edge coloring is conservative if it satisfies two conditions: Assume that $e_1, \ldots, e_m$ are the edges encountered when traveling around a closed cycle. Let $\epsilon_i = 1$ if $e_i$ is traveled in the preferred direction; otherwise $\epsilon_i = -1$. Let $\alpha_1, \ldots, \alpha_m$ be the colors assigned to $e_1, \ldots, e_m$.

1. **Cycle condition:**

   $$\sum \epsilon_i \alpha_i = 0 \quad (3.1)$$

   Assume that $e_1, \ldots, e_n$ are the edges incident to a vertex $v$. Let $\eta_i = 1$ if $v$ is the terminal vertex of $e_i$; otherwise, $\eta_i = -1$. Let $\alpha_1, \ldots, \alpha_n$ be the colors assigned to $e_1, \ldots, e_n$.

2. **Kirchhoff vertex condition:**

   $$\sum \eta_i \alpha_i = 0 \quad (3.2)$$

   An element $\beta = \alpha Q$ of the cut space $W$ assigns to an edge directed from $v_i$ to $v_j$ the color $\alpha_j - \alpha_i$. Such a coloring clearly satisfies the cycle condition. Conversely, suppose $\beta \in F^E$ satisfies the cycle condition. We may assign an arbitrary color to a basing vertex, and extend along a
spanning tree to obtain a unique vertex coloring \( \alpha \) with \( \beta = \alpha Q \). The cycle condition insures that edges not on the spanning tree receive the right colors.

An edge coloring \( \beta \) satisfies the Kirchhoff vertex condition if and only if \( \beta Q^T \) is trivial, that is, \( \beta \in W^\perp \). Thus the conservative edge colorings are precisely the bicycles.

**Remark 3.2.** An equivalent theory of face colorings can also be defined. Compare with [4].

### 4 Plane graphs

By a *plane graph* we mean a graph \( G \) embedded in the plane without accumulation points. Then \( G \) partitions the plane into faces, some of which might be non-compact.

**Definition 4.1.** Let \( G \) be a connected, locally finite plane graph. A *Dehn coloring* of \( G \) is an assignment of colors to the vertices and faces of \( G \) such that a chosen base face is colored 0, and at any edge the conservative coloring condition is satisfied:

\[
\alpha_1 + \beta_1 = \alpha_2 + \beta_2,
\]

where \( \alpha_1, \beta_1, \alpha_2, \beta_2 \) are the colors assigned to vertices and faces \( v_1, R_1, v_2, R_2 \), as in Figure 1.

Like conservative vertex colorings of \( G \), Dehn colorings form a vector space that we will denote by \( C' \). It is not difficult to see that the underlying vertex assignment of any Dehn coloring is a conservative vertex coloring. (If we add the equations (4.1) for edges adjacent to any vertex, then the face colors \( \beta_i \) cancel in pairs.) Let \( r \) denote the restriction from \( C' \) to the vector space \( C \) of conservative vertex colorings.

**Proposition 4.2.** The mapping \( r \) is an isomorphism from \( C' \) to \( C \).

**Proof.** Consider any conservative vertex coloring of \( G \). Beginning at the base face of \( G \), which is colored with 0, “integrate” along any path to another face, a path that is transverse to edges and does not pass through any vertex. If we arrive at an uncolored face \( F \) from a face colored \( \beta \) by crossing an edge with vertices labeled \( \alpha \) and \( \alpha' \), then assign to \( F \) the color \( \alpha - \alpha' + \beta \), where \( \alpha \) is the label on the vertex to the left of the path. Such an assignment is path independent if and only if integrating around any small closed path surrounding a vertex returns the same color with which we began. It is easy to check that this will be the case if and only if condition (2.1) is satisfied at every vertex. In this way we obtain a map \( e : C \to C' \). It is not difficult to see that \( e \) is a homomorphism. That \( r \circ e \) is the identity map on \( C \) is clear. The path-independence of integration implies that \( e \circ r \) is the identity map on \( C' \). \( \square \)

Medial graphs provide a bridge between planar graphs and links. The *medial graph* of a connected locally finite plane graph \( G \) is the plane graph \( M(G) \) obtained from the boundary of a thin regular neighborhood of \( G \) by pinching each edge to create a vertex of degree 4, as in Figure 3.
The medial graph $M(G)$ is the projection in the plane of an alternating link $L$ (possibly with noncompact components if $G$ is infinite) called the medial link. It is well defined up to replacement of every crossing by its opposite. By a component of the medial graph we will mean the projection of a component of $L$.

Remark 4.3. (1) The vector space of Dehn colorings of $G$ is isomorphic to a vector space of Dehn colorings of the link $L$ (see [16] [4]). It is well known that $\mathcal{C}^0$ is isomorphic to the first homology group of the 2-fold cover of $L$ with coefficients in $\mathbb{F}$. Classes of lifts of meridians, one from all but one component of $L$, comprise a basis. While much of our motivation derives from these facts, we do not make explicit use of them here.

(2) When $G$ is finite, $L$ is the Goeritz matrix of the alternating link $L$ associated to $G$. (See [31].)

For the remainder of the section, $\mathbb{F} = GF(2)$. In this case, edge orientations are not needed. Recall that edge colorings correspond bijectively to subsets of $E$.

Given a component of the medial graph $M(G)$, we consider the subset of $E$ consisting of those edges crossed by the component exactly once. Following [22] we refer to the edge set as the residue of the component.

In [26] H. Shank proved that for any finite plane graph $G$ the residues of components of $M(G)$ span $\mathcal{B}$. (Shank ascribed this to J.D. Horton.) An analogous result was shown in [22] for infinite, locally finite plane graphs, but where cycles have a more restrictive definition.

In [29] the second and third authors gave a very short, elementary proof of the following theorem for finite plane graphs $G$ using an idea borrowed from knot theory. The argument here allows for the medial graph to have any number of components, including non-compact components. After finishing the argument, the authors became aware of [2], in which a similar argument is used in the case that $G$ is finite.

**Theorem 4.4.** (cf. Theorem 17.3.5 of [10]) For $\mathbb{F} = GF(2)$, the residues of all but one component of the medial graph $M(G)$ form a basis for the bicycle space $\mathcal{B}$ of the graph $G$.

**Proof.** Regard colorings with 0, 1 of components of $M(G)$ as the elements of a vector space $\mathcal{M}$, with addition and scalar multiplication defined in the obvious way. There is a natural isomorphism...
between $\mathcal{M}$ and the space $\mathcal{C}$ of conservative vertex colorings of $G$. Given an element of $\mathcal{M}$, integrate along paths from the base face in order to color all vertices and faces of $G$. Each time we cross a component of $M(G)$ colored with 1, we change color, until we reach the desired vertex or face. (Here a face is considered to be outside the thin regular neighborhood of $G$ by which we describe $M(G)$, as in Figure 3.) The component can be regarded as a smoothly immersed curve in the plane. By considering intersection numbers modulo 2 (or using the Jordan Curve Theorem) we see that the coloring we obtain is path independent and hence well defined.

The assignment defines a homomorphism from $\mathcal{M}$ to $\mathcal{C}$. An inverse homomorphism is easily defined. Consider any non-crossing point on a component of $M(G)$, and requiring that each be colored with 0.

Like $\mathcal{C}$, the vector space $\mathcal{M}$ has a subspace of constant colorings. Let $\mathcal{M}^0$ denote the quotient space. The isomorphism from $\mathcal{M}$ to $\mathcal{C}$ induces one from $\mathcal{M}^0$ to $\mathcal{C}^0$. It can be made explicit by choosing a base vertex of $G$ near a base component of $M(G)$, and requiring that each be colored with 0.

By Proposition 2.5, $\mathcal{C}^0$ is isomorphic to $\mathcal{B}$, which we identified with the space of conservative edge colorings of $G$. Given a basic element of $\mathcal{M}^0$, a coloring that assigns 1 to a non-base component and 0 to the others, we see that the associated vertex coloring of $G$ assigns different colors to vertices of an edge precisely when the component intersects that edge exactly once. Since these are the edges colored 1 by the associated edge coloring, the proof is complete.

We conclude the section with an example that illustrates most of the ideas so far. Examples involving infinite graphs appear in the next section.

**Example 4.5.** Consider the complete graph $G$ on four vertices embedded in the plane, as in Figure 4. The incidence matrix is:

$$Q = \begin{pmatrix}
-1 & 0 & 1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}$$

while the Laplacian matrix is:

$$L = \begin{pmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3
\end{pmatrix}.$$ 

We choose $v_1$ to be a base vertex. Then the bicycle space $\mathcal{B}$ is isomorphic to the nullspace of the principal minor

$$L_0 = \begin{pmatrix}
3 & -1 & -1 \\
-1 & 3 & -1 \\
-1 & -1 & 3
\end{pmatrix}$$

with rows and columns corresponding to $v_2, v_3, v_4$. Since the determinant of $L_0$ is 16, $\mathcal{B}$ is trivial unless $F$ has characteristic 2.

When $F = GF(2)$, the space $\mathcal{C}^0$ of based conservative vertex colorings is the nullspace of $L_0$, which has basis $(1, 1, 0), (0, 1, 1) \in V^3$. Recall that each of these basis vectors represents a coloring of the vertices of $G$ using 0, 1 with the chosen base vertex $v_1$ receiving 0. A basis vector for $\mathcal{B}$ is obtained from each by selecting the edges that have differently colored vertices. This
Figure 4: Plane Graph $G$ with Vertices and Oriented Edges Labeled

gives $(1, 0, 1, 0, 1, 1), (1, 1, 0, 1, 0, 1) \in E^6$. (Alternatively, the first vector is the sum of the second and fourth rows of $Q$ while the second vector is the sum of the third and fourth.) The residues correspond to two components of the medial link $L$.

5 Graphs with free $\mathbb{Z}$-symmetry

When $G$ is an infinite graph, calculations can still be done provided that the graph has sufficient symmetry.

A graph $G$ has free $\mathbb{Z}$-symmetry if $\mathbb{Z}$ acts freely on $G$ by automorphisms. Any such graph arises by starting with a finite graph $G$ with vertex set embedded in an annulus $A$ (we allow the edges to cross one another), and then lifting to the universal cover of $A$. If $G$ is not contained in any disk in $A$, then $G$ will be connected.

We regard $\mathbb{Z}$ as a multiplicative group with generator $x$. Let $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$ be the vertex and edge sets of $G$. Choose a lift $v_{i,0}$ of each $v_i \in V$ and also a lift $e_{j,0}$ of each $e_j \in E$. We denote the images of $v_{i,0}$ and $e_{j,0}$ under the group element $x^\nu \in \mathbb{Z}$ by $v_{i,\nu}$ and $e_{j,\nu}$, respectively. Then the vertex and edge sets $V, E$ of $G$ each consists of a finite family of vertices $v_{i,\nu}$ and $e_{j,\nu}$, respectively.

Let $F[x^{\pm 1}]$ denote the ring of Laurent polynomials in variable $x$ with coefficients in $F$. The space $C$ is the dual $\text{Hom}(C, F)$ of a finitely generated $F[x^{\pm 1}]$-module $C$ with relation matrix $L = L(x)$. The matrix is finite with coefficients in $F[x^{\pm 1}]$.

For $k \geq 0$, let $\Delta_k(x)$ denote the $k$th elementary divisor of $C$ (abbreviated by $\Delta_k$). It is the greatest common divisor of the determinants of all $(n-k) \times (n-k)$ minors of the Laplacian matrix $L$. It is well defined up to multiplication by units in $F[x^{\pm 1}]$. We call $\Delta_k$ the $k$th Laplacian polynomial of $G$. The polynomial $\Delta_0$ can be regarded as the determinant of the Laplacian matrix of $G$ with additional, homological information about algebraic winding numbers of cycles in the annulus $A$ (see [8], [17]).

The degree of a nonzero Laurent polynomial $f \in F[x^{\pm 1}]$ is the difference of the maximum and minimum degrees of a nonzero monomial in $f$, denoted by $\text{deg}_f f$.

A polynomial $f \in F[x^{\pm 1}]$ is reciprocal if $f(x^{-1})x^n = f(x)$ for some $n \in \mathbb{Z}$. Less formally, $f$ is reciprocal if its sequence of coefficients is palindromic.

Proposition 5.1. Let $G$ be a locally finite graph with free $\mathbb{Z}$-symmetry.

(1) For any $k \geq 0$, the Laplacian polynomial $\Delta_k$ is reciprocal.
\( (2) \Delta_0(G) \) is divisible by \((x - 1)^2\).

**Proof.** (1) After multiplying each row of \( L \) by a suitable unit of \( \mathbb{F}[x^\pm 1] \), we can assume without loss of generality that, for each \( i \), the contribution of the diagonal matrix \( D \) is a constant. From the form of the adjacency matrix it follows that \( L(x^{-1}) = L(x)^T \). Hence \( \Delta_k(x) \) is reciprocal.

(2) If \( \Delta_0 \) is identically zero, then there is nothing to prove. Assume that \( \Delta_0 \) is nonzero. Setting \( x = 1 \) makes the row sums of \( L \) zero. Hence \( \Delta_0(1) = 0 \).

By (1), we can normalize so \( \Delta_0(x) = \Delta_0(x^{-1}) \). Thus \( \Delta_0 \) is reciprocal of even degree, and its roots come in reciprocal pairs. Hence 1 is a double root. \( \square \)

**Remark 5.2.** If \( \overline{G} \) is embedded, then by the medial graph construction, we obtain an alternating link \( \ell = \ell_1 \cup \cdots \cup \ell_\mu \) in the thickened annulus, which we may regard as a solid torus \( S^1 \times D^2 \). Consider the encircled link \( \ell \cup m \), where \( m \) is a meridian \( \{ p \} \times \partial D^2 \), where \( p \in S^1 \). It is not difficult to show that the Laplacian polynomial \( \Delta_0 \) is equal to \((x - 1)\Delta(-1, \ldots, -1, x)\), where \( \Delta \) is the Alexander polynomial of \( \ell \cup m \) with \( \mu + 1 \) variables, and \( x \) is the variable corresponding to \( m \). We will not make use of this fact here.

Now assume that the graph \( \overline{G} \) is embedded in the annulus. Then \( G \) will be planar. Recall that the medial graph \( M(G) \) is the projection of an alternating link \( \mathcal{L} \). Components of \( \mathcal{L} \) are called components of the medial graph.

An **annular cut set** of \( \overline{G} \) is a set of vertices such that when the vertices and incident edges are removed, the resulting graph is contained in a disk neighborhood. The **annular connectivity** \( \kappa(\overline{G}) \) is the minimal cardinality of an annular cut set of \( \overline{G} \).

**Theorem 5.3.** Let \( G \) be a locally finite graph embedded in the plane with free \( \mathbb{Z} \)-symmetry.

(1) For \( \mathbb{F} = GF(2) \), \( \deg_{\mathbb{F}} \Delta_s \) is equal to twice the number of noncompact components of \( M(G) \), where \( \Delta_s \) is the first nonzero Laplacian polynomial of \( G \).

(2) For \( \mathbb{F} = \mathbb{Q} \), \( \deg_{\mathbb{F}} \Delta_0 = \kappa(\overline{G}) \). [DOES this require planarity?]

**Proof.** (1) Recall that the components of the medial graph \( M(G) \) correspond to a basis for the vector space \( \mathcal{C} \) over \( \mathbb{F} = GF(2) \). If \( M(G) \) has closed components, then each \( \mathbb{Z} \)-orbit of these spans a free \( \mathbb{F}[x^\pm 1] \)-summand of the module \( \mathcal{C} \). The module \( \mathcal{C} \) decomposes as \( \text{Tor}(\mathcal{C}) \oplus \mathbb{F}[x^\pm 1]^s \), where \( \text{Tor}(\mathcal{C}) \) is the \( \Lambda \)-torsion submodule of \( \mathcal{C} \), and \( s \) is the number of \( \mathbb{Z} \)-orbits of closed components of \( M(G) \). The first non-vanishing Laplacian polynomial is \( \Delta_s(x) \). Its degree, which is the dimension of \( \text{Tor}(\mathcal{C}) \) regarded as a vector space over \( \mathbb{F} \), is the number of non-compact components of \( M(G) \).

(2) A combinatorial expression for \( \Delta_0 \) was given by R. Forman [8]. Another proof was later given by R. Kenyon [17] [18], and we use his terminology.

\[
\Delta_0(x) = \sum_{k=1}^{\infty} C_k(2 - x - x^{-1})^k. \quad (5.1)
\]

Here \( C_k \) is the number of cycle rooted spanning forests in the graph \( \overline{G} \) having \( k \) components and every cycle essential (that is, non-contractible in the annulus). A **cycle rooted spanning forest** (CRSF) is a subset of edges of \( \overline{G} \) such that (i) every vertex is an endpoint of some edge; and (ii) each component has exactly one cycle.

It follows that the degree of \( \Delta_0 \) is the maximum cardinality \( N \) of a set of pairwise disjoint essential cycles in \( A \). We complete the proof by showing that \( N \) is the annular connectivity \( \kappa(\overline{G}) \).

Regard the annulus \( A \) as lying in the plane, having inner and outer boundary components. Let \( \gamma_1, \ldots, \gamma_n \) be a set of pairwise disjoint essential cycles of maximal cardinality, ordered so that \( \gamma_i \)
lies inside $\gamma_{i+1}$. There is a path with interior in the complement of $\overline{G}$ from the inner boundary to some vertex $v_i$ of $\gamma_1$, since otherwise the obstructing edges would form an essential cycle disjoint from $\gamma_1, \ldots, \gamma_n$. Inductively, there is a path from $v_i$ on $\gamma_i$ to a vertex $v_{i+1}$ on $\gamma_{i+1}$, and from $v_n$ to the outer boundary, with interior in the complement of $\overline{G}$. Then $\{v_{1,0}, \ldots, v_{n,0}\}$ is a vertex cut set for the graph $G$. No vertex cut set can have smaller cardinality, since its projection must have a vertex on each of the cycles $\gamma_1, \ldots, \gamma_n$.

If the annular connectivity $\kappa(\overline{G})$ is equal to 1 then $\overline{G}$ can be split at a vertex $v$, producing a graph $H$ with vertices $v', v''$ so that $G$ is an infinite join of copies $(H_\nu, v'_\nu, v''_\nu)$:

$$\cdots \ast H_\nu \ast H_{\nu+1} \ast \cdots,$$

where $H_\nu$ is joined to $H_{\nu+1}$ along $v'_\nu \in H_\nu$ and $v''_{\nu+1} \in H_{\nu+1}$.

**Corollary 5.4.** If $\kappa(\overline{G}) = 1$, then the Laplacian polynomial $\Delta_0(x)$ of $G$ is equal to $m(x - 1)^2$, where $m$ is the number of spanning trees in $H$.

**Proof.** Note that up to unit multiplication in $\mathbb{Q}[t^\pm 1]$, the term $(x - 1)^2$ is equal to $(2 - x - x^{-1})$. In the proof of Theorem 5.3, $C_k = 0$ unless $k = 1$. Every CRSF with an essential cycle becomes a spanning tree for $H$ when $\overline{G}$ is split along $v$ to produce $H$. Conversely, every spanning tree for $H$ becomes a CRSF with essential cycle when $v'$ and $v''$ are rejoined. Hence there is a bijection between the set of CRSFs with an essential cycle and the set of spanning trees for $H$. 

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**Example 5.5.** Consider the graph $\overline{G}$ embedded in the annulus as in Figure 5. It lifts to the “ladder graph” $G$, which appears in Figure 6. The Laplacian matrix is

$$L = \begin{pmatrix} 3 - x - x^{-1} & -1 \\ -1 & 3 - x - x^{-1} \end{pmatrix} \quad (5.2)$$

The 0th Laplacian polynomial $\Delta_0(x)$ is $(x - 1)^2(x^2 - 4x + 1)$ with $F = \mathbb{Q}$ or $F = GF(2)$. The reader can check that $\{v_1, v_2\}$ is an annular cut set of $\overline{G}$ with minimal cardinality, and $M(G)$ has four components. The bicycle space $B$ has dimension 3 for any field $F$.

**Example 5.6.** Consider the “girder graph” $G$ in Figure 7. The Laplacian matrix is

$$L = \begin{pmatrix} 6 - 2x - 2x^{-1} & -1 - x^{-1} \\ -1 - x & 6 - 2x - 2x^{-1} \end{pmatrix} \quad (5.3)$$

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"Figure 5: Graph $\overline{G}$"  
"Figure 6: Graph $G$"
With $\mathbb{F} = \mathbb{Q}$ the 0th Laplacian polynomial is $\Delta_0(x) = (x-1)^2(4x^2-17x+4)$. Again one can check that the quotient graph has an annular cut set of cardinality 2 but no cut set of smaller size.

If $\mathbb{F} = GF(2)$, then $\Delta_0(x) = (x-1)^2$. It is an amusing exercise to verify that $M(G)$ has exactly two components.

The bicycle space $\mathcal{B}$ has dimension 3 whenever the characteristic of $\mathbb{F}$ is different from 2. When the characteristic is 2, $\mathcal{B}$ is 1-dimensional.

6 Graphs with free $\mathbb{Z}^2$-symmetry

A finite graph $\overline{G}$ with vertex set embedded in the torus lifts to a graph $G$ with a $\mathbb{Z}^2$-action by automorphisms. We regard $\mathbb{Z}^2$ as a multiplicative group with generators $x, y$ corresponding to a fixed meridian and longitude of the torus. Each vertex $v$ of $G$ is covered by a countable collection $v(i,j)$ of vertices such that the action of $x^m y^n$ sends each $v(i,j)$ to $v(i+m,j+n)$, for $m, n \in \mathbb{Z}$. We will assume that $G$ is connected.

As in the case of graphs with free $\mathbb{Z}$-symmetries, the space $C$ of conservative vertex colorings of $G$ is the dual Hom($C, \mathbb{F}$) of a finitely generated module $C$, but here the ring is $\mathbb{F}[x^\pm 1, y^\pm 1]$. The Laplacian matrix $L$ is a relation matrix for $C$, and there is a sequence $\Delta_k(x, y)$ of $k$th elementary divisors (abbreviated by $\Delta_k$), well defined up to multiplication by units. Again, we call $\Delta_k$ the $k$th Laplacian polynomial of $G$.

Example 6.1. Consider the simplest graph $\overline{G}$, having a single vertex $v$, embedded in the torus such that each face is contractible, as in Figure 8. It lifts to a plane graph $G$ with vertices $v(i,j)$, as in Figure 9. The Laplacian matrix is a $1 \times 1$-matrix:

$$L = (4-x-x^{-1}-y-y^{-1})$$

and so the 0th Laplacian polynomial is $\Delta_0(x, y) = 4-x-x^{-1}-y-y^{-1}$. The bicycle space is infinite-dimensional for any field $\mathbb{F}$.

Example 6.2. Consider the Mitsubishi (three diamond) graph $G$ with vertices labeled as in Figure 10. The Laplacian matrix is

$$L = \begin{pmatrix}
6 & -1-x^{-1}-y^{-1} & -1-y^{-1}-xy^{-1} \\
-1-x-y & 3 & 0 \\
-1-y-x^{-1}y & 0 & 3
\end{pmatrix}.$$  

(6.2)

The Laplacian polynomial is $\Delta_0(x, y) = 6(6-x-x^{-1}-y-y^{-1}-xy^{-1}-x^{-1}y)$. The polynomial vanishes modulo 2 since the medial graph $M(G)$ has closed components.
7 Mahler measure and spanning trees

We review some basic notions of algebraic dynamics as applied to locally finite graphs. A general treatment can be found in [7] or [25].

Assume that \( G \) is a connected locally finite graph, not necessarily planar, that admits a free \( \mathbb{Z}^d \)-action by automorphisms. We assume that the vertices and edges of \( G \) consist of the orbits of finitely many vertices \( v_1, \ldots, v_n \) and edges \( e_1, \ldots, e_m \), respectively. For any \( x \in \mathbb{Z}^d \), we denote the vertex \( (x_1 \ldots x_d)v_i \) by \( v_i,x \), where \( x = (s_1, \ldots, s_d) \). We use similar notation for edges.

Let \( \overline{G} \) be the quotient graph of \( G \). By abuse of notation, we denote its vertices by \( v_1, \ldots, v_n \) and edges by \( e_1, \ldots, e_m \). We regard \( \overline{G} \) as a covering graph of \( G \). If \( \Lambda \subset \mathbb{Z}^d \) is any subgroup, then \( \Lambda \)-intermediate covering graph of \( G \). If \( \Lambda \) has finite index \( r \), then \( G_\Lambda \) is a finite, \( r \)-sheeted covering graph.

The vector space \( C \) of conservative vertex colorings is the dual space \( \text{Hom}(C, \mathbb{F}) \), where \( C \) is a finitely generated \( \Lambda \)-module with presentation matrix \( L \). We regarded \( L \) as a matrix over \( \mathbb{F}[x_1^\pm 1, \ldots, x_d^\pm 1] \). However, we can also regard it over the ring \( \mathcal{R}_d = \mathbb{Z}[x_1^\pm 1, \ldots, x_d^\pm 1] \) of Laurent polynomials in \( d \) variables over the integers. By our assumptions, \( C \) is a finitely generated \( \mathcal{R}_d \)-module.

We replace the field \( \mathbb{F} \) with the additive circle group \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \). Then \( \text{Hom}(C, \mathbb{T}) \) is the Pontryagin dual \( \hat{C} \) of \( C \). A homomorphism is a function \( \rho \) that assigns a “color” \( \alpha_{i,s} \in \mathbb{T} \) to each vertex \( v_i,s \) in such a way that, when extended linearly, all \( \mathbb{Z}^d \)-multiples of row-vectors of \( L \) are mapped to zero. Clearly, \( \hat{C} \) is an abelian group under coordinate-wise addition.

We endow \( C \) with the discrete topology, and the space of homomorphisms with the compact-open topology. Then \( \hat{C} \) is a compact, 0-dimensional topological group. Moreover, the module actions of \( x_1, \ldots, x_d \) determine commuting homeomorphisms \( \sigma_1, \ldots, \sigma_d \) of \( \hat{C} \). Explicitly, if \( \rho \) assigns \( \alpha_{i,s} \) to \( v_i,s \), then \( \sigma_j \rho \) assigns \( \alpha_{i,s'} \), where \( s' \) is obtained by adding 1 to the \( j \)th component of \( s \). Consequently, \( \hat{C} \) has a \( \mathbb{Z}^d \)-action \( \sigma : \mathbb{Z}^d \to \text{Aut}(\hat{C}) \). We denote \( \sigma(s) \) by \( \sigma_s \).

**Definition 7.1.** Let \( \Lambda \subset \mathbb{Z}^d \) be a subgroup. A \( \Lambda \)-periodic point of \( \hat{C} \) is homomorphism \( \rho \) such that \( \sigma_s \rho = \rho \), for any \( s \in \Lambda \).

The set of all \( \Lambda \)-periodic points is a subgroup of \( \hat{C} \), denoted here by \( \text{Per}_\Lambda(\sigma) \).
Proposition 7.2. Let $\Lambda \subset \mathbb{Z}^d$ be a subgroup. Then $\text{Per}_\Lambda(\sigma)$ is isomorphic to the group of conservative vertex colorings of $G_\Lambda$. If $\Lambda$ has finite index, then $\text{Per}_\Lambda(\sigma)$ consists of $\tau(G_\Lambda)$ circles, where $\tau(G_\Lambda)$ is the number of spanning trees of $G_\Lambda$.

Proof. There is a natural isomorphism, which is also a homeomorphism, between $\text{Per}_\Lambda(\sigma)$ and the group of conservative vertex colorings of $G_\Lambda$. The latter is a subspace of $\mathbb{T}^{V_\Lambda}$, where $V_\Lambda$ is the vertex set of $G_\Lambda$. It can be computed from the Laplacian matrix of $G_\Lambda$. The corank of the matrix is 1. By Kirchhoff’s matrix-tree theorem (see, for example, Chapter 13 of [10]), the absolute value of the determinant of any submatrix obtained by deleting a row and column is equal to $\tau(G_\Lambda)$. Hence the conservative colorings of $G_\Lambda$ consist of $\tau(G_\Lambda)$ pairwise disjoint circles. \hfill \Box

The topological entropy $h(\sigma)$ of our $\mathbb{Z}^d$-action $\sigma$ is a measure of complexity. The general definition can be found in [7] or [25]. By a fundamental result D. Lind, K. Schmidt and T. Ward, [20] [25], it is equal to the exponential growth rate of $|\text{Per}_\Lambda(\sigma)|$, the number of components of $\text{Per}_\Lambda(\sigma)$, using a suitable sequence of subgroups $\Lambda$.

$$h(\sigma) = \limsup_{\langle \Lambda \rangle \to \infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \log |\text{Per}_\Lambda(\sigma)|.$$  \hfill (7.1)

Here $\langle \Lambda \rangle$ is the minimum length of a nonzero element of $\Lambda$. Heuristically, the condition that $\langle \Lambda \rangle$ tends to infinity ensures that the sublattice $\Lambda$ of $\mathbb{Z}^d$ grows in all directions as we take a limit.

There is a second way to compute $h(\sigma)$, which uses Mahler measure.

Definition 7.3. The logarithmic Mahler measure of a nonzero polynomial $f(x_1, \ldots, x_d) \in \mathcal{R}_d$ is

$$m(f) = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_d})| d\theta_1 \cdots d\theta_d.$$  \hfill (7.2)

Remark 7.4. (1) The integral in Definition 7.3 can be singular, but nevertheless it converges.
(2) When \( d = 1 \), Jensen’s formula shows that \( m(f) \) can be described another way. If \( f(x) = c_s x^s + \cdots c_1 x + c_0, c_0 c_s \neq 0 \), then
\[
m(f) = \log |c_s| + \sum_{i=1}^{s} \log |\lambda_i|,
\]
where \( \lambda_1, \ldots, \lambda_s \) are the roots of \( f \).

(3) Logarithmic Mahler measure arises as topological entropy in many situations (see [25]). Nevertheless, \( \exp m(f) \) is also useful, and it is called Mahler measure of \( f \) and is denoted by \( M(f) \).

(4) If \( f, g \in \mathcal{R}_d \), then \( m(fg) = m(f) + m(g) \). Moreover, \( m(f) = 0 \) if and only if \( f \) is the product of 1-variable cyclotomic polynomials, each evaluated at a monomial of \( \mathcal{R}_d \) (see [25]).

By [25] (see Example 18.7), \( h(\sigma) \) is equal to the logarithmic Mahler measure \( m(\Delta_0) \) of the 0th Laplacian polynomial of \( G \). By Proposition 7.2, we have:

**Theorem 7.5.** Let \( G \) be a locally finite graph with free \( \mathbb{Z}^d \) action by automorphisms and finite quotient graph \( \overline{G} \). Then
\[
m(\Delta_0) = \limsup_{(\Lambda) \to \infty} \frac{1}{|\mathbb{Z}^d/\Lambda|} \log \tau(\Lambda),
\]
where \( \tau(\Lambda) \) is the number of spanning trees of the covering graph \( \overline{G} \).

**Proposition 7.6.** (Cf. [5]) Let \( G \) be a locally finite graph with free \( \mathbb{Z}^d \) action by automorphisms and finite quotient graph \( \overline{G} \). Then
\[
m(\Delta_0) \leq |V| \log \frac{2|E|}{|V|},
\]
where \( V, E \) are the vertex and edge sets, respectively, of \( \overline{G} \).

**Proof.** By a result of G.R. Grimmett [13],
\[
\tau(\Lambda) \leq \frac{1}{|V|} \left( \frac{2|E|}{|V| - 1} \right)^{|V| - 1},
\]
for every finite-index sublattice \( \Lambda \) of \( \mathbb{Z}^d \). Letting \( r = |\mathbb{Z}^d/\Lambda| \), we have
\[
m(\Delta_0) \leq \limsup_{r \to \infty} \frac{1}{r} \log \frac{1}{r|V|} \left( \frac{2r|E|}{r|V| - 1} \right)^{r|V| - 1}.
\]
The result follows by elementary analysis.

Partition functions and other analytic tools have been used to compute growth rates of the number of spanning trees. See, for example, [32], [27], [6] and [30]. The growth rates obtained are known as “bulk limits” or “thermodynamic limits.” These investigations consider certain finite subgraphs of \( G \) rather than finite quotients, as done here. We explain how our algebraic approach gives the same limits, if suitable normalization is applied. The fact that the thermodynamic limits for many examples are Mahler measures was previously noted in [14].

Let \( \Lambda \) be a finite-index subgroup of \( \mathbb{Z}^d \) and let \( R \) a fundamental domain. In the literature, \( R \) is usually chosen to be a cube. Let \( G|_R \) denote the full subgraph on the vertices with indices in \( R \).
The number of such vertices is $s = k|\mathbb{Z}^d/\Lambda \mathbb{Z}^d|$, where $k$ is the number of vertex orbits of $G$. Taking a limit over increasingly large domains $R$, one defines the thermodynamic limit (or spanning tree constant)

$$
\lambda_G = \lim_{s \to \infty} \frac{1}{s} \log \tau(G|R),
$$

where $\tau$ is the number of spanning trees.

Since every spanning tree of $H$ can be viewed as a spanning tree for $G\Lambda$, we see immediately from our results above that $\lambda_G \leq \frac{1}{k} m(\Delta_0)$. In fact, a fairly standard argument shows that equality holds.

Let $T$ be a spanning tree for $G\Lambda$. Its restriction $T_R$ to $G|R$ might not be connected. However, when $d > 1$, within a slightly larger domain $R'$ containing $R$ we can extend $T_R$ to a spanning tree $T'$. We take $R'$ to consist of elements of $\mathbb{Z}^d$ that are some bounded distance from $R$, so that every edge of $G$ with a vertex in $R$ has its other vertex in $R'$, and the graph $G|R\setminus R$ is connected. (This is where $d > 1$ is needed.) Then the number $s'$ of vertices of $R'$ satisfies $s'/s \to 1$ as $s \to \infty$. We can choose $T'$ to contain $T_R$, thereby ensuring that $T \mapsto T'$ is an injection, since $T$ can be recovered from $T|R'$. This gives the reverse inequality $\lambda_G \geq \frac{1}{k} m(\Delta_0)$.

In the case $d = 1$, $G|R\setminus R$ consists of two connected components. Applying the above construction, we can obtain a graph $T'$ that is either a spanning tree or a two-component spanning forest for $G|R'$. Any such forest can be obtained from a spanning tree by deleting one of the $s' - 1$ edges. Hence $\tau(G\Lambda)$ is no more than $s'$ times the number of spanning trees of the graph on $R'$. This rough upper bound suffices to give the desired growth rate.

We have proved:

**Theorem 7.7.** The thermodynamics limit $\lambda_G$ is equal to $\frac{1}{k} m(\Delta_0)$, where $k$ is the number of vertex families in $G$.

**Example 7.8.** For the ladder graph in Example 5.5,

$$
m(\Delta_0(x)) = m(x^2 - 4x + 1) = \log(2 + \sqrt{3}) \approx 1.317.
$$

The upper bound of Proposition 7.4 (with $|V| = 2, |E| = 3$) is 2.198. The thermodynamic limit is approximately 0.658 [27]. Note that this agrees with $\frac{1}{3} \log(2 + \sqrt{3}) \approx 0.658$ although $d = 1$.

**Example 7.9.** For the grid in Example 6.1, $m(\Delta_0(x,y))$ can be estimated numerically. In this case, our growth rate agrees with the thermodynamic limit. Both are 1.166. The upper bound of Proposition 7.4 (with $|V| = 1, |E| = 2$) is 1.386.

**Example 7.10.** A circulant graph $C^{s_1,\ldots,s_k}_n$ is a $2k$-regular graph with $n$ vertices $v_1, \ldots, v_n$ such that $v_i$ is adjacent to $2k$ vertices $v_{i+s_1}, \ldots, v_{i+s_k}$, where indices are taken modulo $n$. Several authors [21], [12] have investigated the growth rate $\kappa(C^{s_1,\ldots,s_k}_n)$ using a blend of combinatorics and analysis. We recover the growth rates very quickly with algebraic methods.

Let $\Lambda = (n) \subset \mathbb{Z}$. The graph $C^{s_1,\ldots,s_k}_n$ can be regarded as an $n$-sheeted cover $G\Lambda$ of a graph $\overline{G}$ with a single vertex $v$ and $2k$ edges immersed in the annulus. The $j$th edge winds $s_j$ times. A simple example appears in Figures 11 and 12 below.

It is immediate that the Laplacian polynomial is

$$
\Delta_0(x) = 2k - \sum_{j=1}^k (x^{s_j} + x^{-s_j}).
$$
Figure 11: Circulant Graph $C_5^{1,2}$

Figure 12: Circulant Cover $G$ and Quotient $\overline{G}$
The growth rate $\lim_{n \to \infty} \frac{1}{n} \log \kappa(C_n^{s_1, \ldots, s_k})$ is equal to the logarithmic Mahler measure $m(\Delta_0)$. When $s_1 = 1$, $s_2 = 2$ and $s_j = 0$ for $j > 2$, as in Figure 11, $\Delta_0(x) = 4 - x - x^{-1} - x^2 - x^{-2} = (x - 1)^2(x^2 + 3x + 1)$ and $\lim_{n \to \infty} \frac{1}{n} \kappa(C_n) = (3 + \sqrt{5})/2$, as obtained in [12] (see p. 795).

In general,

$$
\lim_{n \to \infty} \frac{1}{n} \log \kappa(C_n^{s_1, \ldots, s_k}) = m(\Delta_0(x)) = \int_0^1 \log \left( 2k - \sum_{j=1}^k (x^{2\pi i s_j} + x^{-2\pi i s_j}) \right) dx
$$

$$
= \int_0^1 \left( \log \sum_{j=1}^k (2 - 2 \cos 2\pi s_j x) \right) dx
$$

$$
= \int_0^1 \left( \log 4 \sum_{j=1}^k \sin^2 \pi s_j x \right) dx
$$

$$
= (\log 4) \left[ \int_0^1 \log \left( \sum_{j=1}^k \sin^2 \pi s_j x \right) dx \right],
$$

which agrees with Lemma 2 [12].

We conclude with a comment about Theorem 6 [12], which states:

$$
\lim_{s_k \to \infty} \ldots \lim_{s_1 \to \infty} \lim_{n \to \infty} \frac{1}{n} \log \kappa(C_n^{s_1, \ldots, s_k}) = (\log 4) \left[ \int_0^1 \cdots \int_0^1 \log \left( \sum_{j=1}^k \sin^2 \pi x_j \right) dx_1 \cdots dx_k \right].
$$

In view of Definition 7.3 this quantity is simply the Mahler measure of the $k$-variable polynomial that is obtained from $\Delta_0(x)$ by replacing each $x^{s_j} + x^{-s_j}$ with $x_j + x_j^{-1}$.

**References**


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