Toeplitz Minimal Flows Which Are Not Uniquely Ergodic

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Summary. We study minimal symbolic dynamical systems which are orbit closures of Toeplitz sequences. We construct 0–1 subshifts of this type for which the set of ergodic invariant measures has any given finite cardinality, is countably infinite or has cardinality of the continuum.

The first example of a minimal flow which is not uniquely ergodic was found by Markov (cf. Nemytskii and Stepanov, 1960, p. 312). A paper of Oxtoby (1952) includes a particularly elegant example of such a flow, obtained as the orbit closure of a point in \([0,1]^2\). Later Jacobs and Keane (1969) defined a class of almost periodic 0–1 sequences, called Toeplitz sequences, which includes Oxtoby’s sequence. Although the orbit closure of a regular Toeplitz sequence (see Sect. 2) is always uniquely ergodic, Markley and Paul (1979) have shown that in a certain sense most non-regular Toeplitz sequences yield minimal flows which are not uniquely ergodic.

We consider the problem of describing the invariant measures on the orbit closure of a Toeplitz sequence. We generalize the definition of Toeplitz sequences to sequences in a compact symbol space \(\Sigma\). In Sect. 2 we identify the maximal equicontinuous factor of the flow; this was done by Eberlein (1970) for regular Toeplitz 0–1 sequences. In Sect. 3, we determine the ergodic measures for Oxtoby’s flow; there are exactly two. We construct analogous flows in \(\Sigma^n\) for which the set of ergodic measures has the same cardinality as \(\Sigma\). In Sect. 4 we construct Toeplitz 0–1 sequences for which the orbit closure has the measure structure of a skew product, with the maximal equicontinuous factor as base and a freely chosen subshift of \([0,1]^2\) as fiber. This construction yields minimal flows for which the set of ergodic measures has any given finite cardinality, is countably infinite or has cardinality of the continuum.

The last section contains computations of entropy for our examples. We find Toeplitz flows with entropy arbitrarily close to \(\log 2\). Markley and Paul (1979) show that a non-regular Toeplitz flow “usually” has positive entropy.
The results of this paper are contained in the author’s thesis written at Yale University. I am deeply grateful to Shlomo Kacznabi for his direction and teaching. I also wish to thank John Oxtoby for his help and encouragement.

1. Preliminaries

We summarize some basic definitions and results; we refer the reader to Oxtoby (1952) and Ellis (1969) for more details.

By flow we will mean a pair \( (X, T) \) where \( X \) is a compact metrizable space and \( T \) is a homeomorphism of \( X \) to itself. \( (X, T) \) is minimal if \( X \) has no proper closed \( T \)-invariant subset. \( \mathbb{B}(X) \) will denote the \( \sigma \)-algebra of Borel sets of \( X \). An invariant measure \( \mu \) for \( (X, T) \) is a probability measure \( \mu \) on \( \mathbb{B}(X) \) with \( \mu(T^{-1}B) = \mu(B) \) for all \( B \in \mathbb{B}(X) \); the measure is ergodic if every \( T \)-invariant Borel set has measure 0 or 1. The invariant measures for \( (X, T) \) form a non-empty closed set, and the ergodic measures are exactly the extreme points of this set. The flows are uniquely ergodic if it admits only one (ergodic) invariant measure.

A compact topological group \( G \) is monothetic if some \( g \in G \) generates a dense subgroup of \( G \); \( g \) is called a (topological) generator of \( G \). \( G \) is necessarily abelian. We also denote by \( g \) the translation \( h \mapsto h + g \) on \( G \). Then \( (G, g) \) is a minimal flow, and the Haar measure on \( G \) is the unique invariant measure.

The flow \( (Y, S) \) is a factor of \( (X, T) \) if there is a continuous map \( \pi \) of \( X \) onto \( Y \), with \( \pi \) a homeomorphism when \( X = (X, T) \) and \( Y, S \) are isomorphic as flows. Every minimal flow \( (X, T) \) has a maximal equicontinuous factor (Ellis and Gottschalk, 1960). This can be characterized (up to flow isomorphism) as a factor \( \pi: (X, T) \to (G, g) \), where \( G \) is a compact metrizable monothetic group with generator \( g \), such that for any other such factor \( \xi: (X, T) \to (G', g') \) we have a factor map \( \psi: (G, g) \to (G', g') \) with \( \psi \circ \pi = \xi \).

The importance of the maximal equicontinuous factor to the problem of determining the invariant measures on \( (X, T) \) can be seen as follows: if \( \mu \) is an invariant measure on \( (X, T) \) then \( \mu \circ \pi^{-1} \) is an invariant measure on \( (G, g) \), so it must be equal to the Haar measure \( m \). If \( B \in \mathbb{B}(G) \), then \( \mu(B) = m(B) \circ \pi^{-1}(B) = m(B) \). Thus the invariant measures on \( (X, T) \) all coincide on the \( \sigma \)-algebra \( \pi^{-1}(\mathbb{B}(G)) = \mathbb{B}(X) \).

We will use the following fact from Paul (1976).

**Proposition 1.1.** Let \( (X, T) \) be a minimal flow and \( \pi: (X, T) \to (G, g) \) a factor map, with \( G \) a compact metrizable monothetic group with generator \( g \). If for some \( x \in X \) we have \( \pi^{-1}(\{x\}) = \{x\} \), then \( (G, g) \) is the maximal equicontinuous factor of \( (X, T) \).

2. Toeplitz Sequences

We will generalize the usual definition of Toeplitz \( 0 \) \( -1 \) sequences to sequences in \( X = \Sigma^\omega \), where \( \Sigma \) is a compact metric space. We write elements of \( X \) as \( x = (x(n)) \). The metric
where \( \phi(e) \) is the residue of \( n \) modulo \( p_i \). We let \( G \) be the inverse limit group, \( G = \lim \mathbb{Z}/p_i \mathbb{Z} \). That is,
\[
G = \{ (n_i) : n_i \equiv n_i \mod p_i \text{ for } i < j \}
\]
and \( (n_i + m_i) = (n_i + m_i) \) where \( n_i + m_i \) is taken modulo \( p_i \). We denote by \( \bar{1} \) the element \( 1 \) in \( G \), and \( \bar{n} = n \cdot \bar{1} \) for \( n \in \mathbb{Z} \). The metric
\[
|a_{n_k} - (m_k)| = \max \left[ \frac{1}{|i+1|}, |n_i + m_i| \right]
\]

gives the usual inverse limit topology on \( G \). \( G \) is a compact monothetic group with generator \( \bar{1} \).

**Theorem 2.2.** \((G, \bar{1})\) is the maximal equicontinuous factor of \((\mathbb{Z}/Z)\).

**Remark.** It can be seen algebraically that \( G \) is independent of the choice of period structure \( p_i \). In fact, \( G \) can be obtained without resorting to period structures as \( \lim \mathbb{Z}/p_i \mathbb{Z} \), where the inverse limit is taken over all \( p \in \mathbb{P}(0) \), respecting the homomorphisms \( \mathbb{Z}/q \mathbb{Z} \rightarrow \mathbb{Z}/p_i \mathbb{Z} \) for \( p_i \) (cf. Jacobson, 1980, p. 72–74).

For each \( i \in \mathbb{N}, n \in \mathbb{Z}/p_i \mathbb{Z} \), we set \( A_i = \{ \mathbb{S}^n : m = n \mod p_i \} \).

**Lemma 2.3.** (i) \( A_i \) is exactly the set of all \( \omega \in \mathbb{E}(0) \) with the same \( p_i \)-skeleton as \( \mathbb{S}^n \).

(ii) \( A_i \) is a partition of \( \mathbb{E}(0) \) into relatively open sets.

(iii) \( A_i = A_i \) for \( i < j \) and \( m = n \mod p_i \).

(iv) \( A_i = A_i \).

**Proof.** Let \( \omega \in A_i \); we will show that \( \omega \) has the same \( p_i \)-skeleton as \( \mathbb{S}^n \). Clearly \( \text{Per}(\omega, \sigma) = \text{Per}(\mathbb{S}^n, \sigma) \) for all \( \sigma \in \Sigma \). Suppose \( k \in \text{Per}(\omega, \sigma) \setminus \text{Per}(\mathbb{S}^n, \sigma) \); then we can find \( k' = k \mod p_i \) and \( r \) with \( k' \in \text{Per}(\mathbb{S}^n, \sigma) \). Then for any \( k' = k \mod p_i \), \( 0 \leq k' < p_i \). Since \( \omega(k') = \sigma \) for all \( k' = k \mod p_i \), we obtain
\[
|a_{k'}(r)| \geq 2^{-p_i} |a_{k'}(r)| \quad \text{for all } a \neq a_{k'}.
\]
This contradicts \( \omega \in A_i \).

A similar argument shows that if \( n = m \mod p_i \), then \( A_i \) and \( A_i \) are a positive distance apart. For \( \mathbb{S}^n \) and \( \mathbb{S}^n \) must have different \( p_i \)-skeletons since \( p_i \) is an essential period of \( n \). We can find \( k \in \text{Per}(\mathbb{S}^n, \sigma) \setminus \text{Per}(\mathbb{S}^n, \sigma) \), with \( i < j \) and \( \sigma = \sigma \). We deduce as above that \( |a_{k'}(r)| \geq 2^{-p_j} |a_{k'}(r)| \) for all \( a \neq a_{k'} \) and \( a \neq a_{k'} \). Since \( A_i \) and \( a \neq a_{k'} \) are a positive distance apart. For \( \mathbb{S}^n \) and \( \mathbb{S}^n \) must have different \( p_i \)-skeletons since \( p_i \) is an essential period of \( n \). We can find \( k \in \text{Per}(\mathbb{S}^n, \sigma) \setminus \text{Per}(\mathbb{S}^n, \sigma) \), with \( i < j \) and \( \sigma = \sigma \). We deduce as above that \( |a_{k'}(r)| \geq 2^{-p_j} |a_{k'}(r)| \) for all \( a \neq a_{k'} \) and \( a \neq a_{k'} \). Since \( A_i \) and \( k \notin \text{Per}(\mathbb{S}^n, \sigma) \), \( A_i \) is a finite partition of \( \mathbb{E}(0) \). (i) is true and the rest of (i) follows. Parts (ii) and (iv) are immediate.

**Proof of Theorem 2.2.** For \( \chi = (n \in \mathbb{E}(0) \) we set
\[
A_{\chi} = \bigcup_{i} A_i,
\]
where \( \chi \) is an invariant measure on \((\mathbb{E}(0), S)\). For \( B \in \mathbb{E}(0) \) we write \( B = (B \cap \mathbb{S}^n) \cup (B \setminus \mathbb{S}^n) \). Since \( n \) is 1–1 on \( \mathbb{S}^n \) and \( B \setminus \mathbb{S}^n \), \( \mu(B \cap \mathbb{S}^n) = m \cdot \mu(B \cap \mathbb{S}^n) = m \cdot \mu(B \setminus \mathbb{S}^n) \). Thus \( \mu(B) = \mu(B \setminus \mathbb{S}^n) \). This determines \( \mu \) uniquely.
3. Oxtoby Sequences

We construct Oxtoby sequences in $\mathbb{Z}^2$ analogous to Oxtoby's example. For $\Sigma = \{0, 1\}$ our construction differs slightly from Oxtoby's; the change was made for notational convenience and does not materially affect any results or proofs.

Construction. Let $(p_i)_{i \in \mathbb{N}}$ be a fixed sequence of natural numbers with $p_i \leq 1$, and $p_i \geq 1$ for all $i \in \mathbb{N}$. Fix a dense sequence $(\pi_i)_{i \in \mathbb{N}}$ in $\Sigma$ with the property that every element of the sequence appears infinitely often in the sequence. (For $\Sigma = \{0, 1, \ldots, r\}$ we may take $\pi_i = i$ mod $r$.)

We define the sequence $\eta \in \mathbb{Z}^2$ by inductive steps. The first step is to set $\eta(0) = \sigma_i$ for all $n = 0 \mod p_i$. For each $k \in \mathbb{Z}$ we set $J(k, 0) = \{k, k + 1, p_i, k + 1, \ldots\}$. Step 2 is to set $\eta(n) = \pi_i$ for all $n \in J(k, 0)$ with $i = -1 \mod p_i$. In general, for $i \in \mathbb{N}$ we set $J(k, i) = \{k, k + p_i, \ldots\}$ for which $\eta(n)$ has not yet been defined at the end of the $i$th step. The $(i + 1)^{th}$ step is to set $\eta(n) = \pi_i$ for $n \in J(k, i)$ with $k = i \mod p_i$.

Sequences $\eta$ defined as above will be called Oxtoby sequencies. Note that after the $i$th step $\eta$ is defined on all of $\{n \mod p_i, 1\}$. The construction is periodic at each step, so $\eta$ is a Oxtoby sequence. For $i < j$ and $\pi_i \in \mathbb{Z}$, $J(i, j)$ is a union of sets of the from $J(k, i)$. $\eta(n)$ is exactly the set on which $\eta$ is defined at the end of the $i$th step, since the sets $J(k, i)$ are translates of one another by multiplies of $p_i$ and are filled with different symbols at different stages of the construction. It is clear that the $p_i$-skeleton is not periodic with any smaller period, so $\eta(n)$ is a period structure for $\eta$.

We continue to use the notation of Sect. 2. In particular, $\preccurlyeq ([\eta], i) \rightarrow (G, \bar{1})$ is the maximal equicontinuous factor.

Proposition 3.1. Oxtoby sequence $\eta$ is regular if and only if

$$\sum_{i \in \mathbb{N}} P_i = \infty$$

Proof. Recall that $d_i = 1 - \left(\pi_i \in \mathbb{Z} : \eta(n) \in \text{Per}_{\pi_i}(\eta)\right)$. Then $d_i = 2/p_i$. For $i \geq 1$,

$$d_{i+1} = d_i + \left(1 - d_i\right) p_i$$

since $p_i$ is the proportion of $(0, p_i, 1) \in \text{Per}_{\pi_i}(\eta)$ which is in $\text{Per}_{\pi_i}(\eta)$. Hence

$$1 - d_{i+1} = \left(1 - d_i\right) \left(1 - \frac{p_i}{P_i}\right)$$

$$= \left(1 - \frac{2}{p_i}\right) \left(1 - \frac{2}{P_i}\right)$$

Theorem 3.2. If the Oxtoby sequence $\eta$ is not regular, then the set of ergodic invariant measures on $(\mathbb{Z}^2, \Sigma)$ is in one-to-one correspondence with $\mathbb{N}$.

Before proving this theorem, we will establish some notation and two lemmas.

Notation. By Corollary 2.4, for each $e \in \mathcal{G}$ the sets $\text{Per}_{\pi_i}(\eta)$ are independent of the choice of $\omega = (\pi_i)^n$. We will sometimes denote these sets $\text{Per}_{\pi_i}(\eta)$ and $\text{Aper}(\eta)$.

Lemma 3.3. (i) For all $e = (\eta, \pi)$, $\omega(n) = \text{Aper}(\eta)$ on $\text{Per}_{\pi_i}(\eta)$ for all $\pi_i = \pi_i \in \mathcal{G}$.

Proof. (i) Let $\eta(n) = e = (\eta, \pi)$. For each $i$, $\omega$ has the same $p_i$-skeleton as $S^n\eta$ and so $\{n \in \mathbb{N} : \omega(n) = \pi_i\} \subset \text{Per}_{\pi_i}(\eta)$. If $S^n\eta = \text{Aper}(\eta)$ then $\text{Per}_{\pi_i}(\eta)$ is constant on $J(i, 0)$. If $\text{Per}_{\pi_i}(\eta)$ is constant on $J(i, 0)$, then $\omega(n)$ is constant on $J(i, 0)$ since $\omega(n)$ is constant on $J(i, 0)$. Hence $\omega(n)$ is constant on $\{n \in \mathbb{N} : \omega(n) = \pi_i\}$ for all $i$. If $\{n \in \mathbb{N} : \omega(n) = \pi_i\}$ is constant so we can conclude that $\omega(n)$ is constant on $\text{Aper}(\eta)$. If either of these conditions fail, it is easy to see that $\omega$ as above for some $\mathbb{Z}$; then $\omega = S^n\eta$ and $\text{Aper}(\eta)$.

(ii) Let $\eta(n) \in \mathcal{G}$ and $\pi_i \in \mathbb{Z}$s. The sequences $S^n\eta$ all have the same $p_i$-skeleton for $i > j$, so for each $\omega = \text{Aper}(\eta)$, $S^n\eta$ is eventually constant on $\{n \in \mathbb{N} : \omega(n) = \pi_i\}$ for all $\pi_i = \pi_i \in \mathcal{G}$. Then $\omega = S^n\eta$ converges to the desired $\omega$ as $n \to \infty$.

Set $Z = G \times \Sigma$, with the product topology. We define a flow $\theta$ on $Z$ for $T(g, \sigma) = (g + 1, \sigma)$. The ergodic measures on $(Z, T)$ are exactly those of the form $\pi = \pi \times \delta$, where $\pi$ is the Haar measure on $G$ and $\delta$ is the point measure on $\Sigma$. $d_{\pi_i}(\pi) = 1$. We define a map $\psi : \theta \rightarrow \mathcal{G}$ by mapping $(g, \sigma)$ to the unique $\omega = \text{Aper}(\eta)$ with $\omega(n) = \pi_i$. The map $\psi = 1 - \theta$ except on $\{\pi_i\}$ and $\theta(T) = S \psi$. It can be seen that $\psi$ is not continuous, but we do have:

Lemma 3.4. $\psi$ is bimeasurable.

Proof. First we show $\psi$ is measurable. Sets of the form

$$U = \{\psi = \pi_i\} = \text{Aper}(\eta) \cap \{0, 1, \ldots, 4\}$$

and their translates by powers of $S$ form a sub-basis for $\mathcal{A}(\text{Th}(\mathcal{G}))$, so it suffices to show $\psi^{-1}(U) \cap \{0, 1, \ldots, 4\}$.

We write $U = (U \cap C) \cup (U \cap \bar{C})$. Recall $C = \{\omega = \text{Aper}(\eta) : \omega \in \mathcal{G}\}$ for $0 = 0, 1, \ldots, 4$. We set

$$U = \{\psi = \pi_i\} = \text{Aper}(\eta) \cap \{0, 1, \ldots, 4\} \cup \{\psi = \pi_i : \pi_i \in \mathcal{G}\}$$
Then

\[ U \cap C = \{(x, y) \in \mathbb{Z}^2 \mid |x| < 1, \ |y| < 1\} \times \mathbb{Z}\]

and so \( \varphi^{-1}(U) = \mathbb{Z} \times \mathbb{Z} \).

To show \( \varphi^{-1} \) is measurable, we note that the sets of the form

\[ W = \{(x, y, z, t) \in A \times C \mid |x|, |y|, |z|, |t| < 1\} \]

generate \( \mathcal{G} \). If we set \( B = U \cap C \), where \( U \) and \( C \) are as above, then

\[ S^k B = \{(x, y, z, t) \in A \times C \mid |x|, |y|, |z|, |t| < 1\} \]

for \( k \geq 0 \).

Proof of Theorem 3.2. For each \( a \in \Xi \) we define \( \mu_a \) on \( \mathcal{G}(\mathcal{E}(a)) \) by

\[ \mu_a(B) = m(B \cap \{x \mid \varphi(x) = a\}) \]

It is easy to see that \( \mu_a \) is an ergodic measure on \( \mathcal{G}(\mathcal{E}(a)) \) with \( \mu_a(G \times \{a\}) = 1 \).

Finally, since \( \mu \) is an ergodic measure on \( \mathcal{G}(\mathcal{E}(a)) \), the formula

\[ \mu(B) = m(B \cap \{x \mid \varphi(x) = a\}) \]

for \( B \in \mathcal{G}(\mathcal{E}(a)) \) is determined by \( \mu \).

Thus \( \mu = \mu_a \).

4. Topelitz Sequences Constructed from Shifts

In Sect. 3 we obtained minimal flows with arbitrarily many ergodic measures at the expense of working with an arbitrary compact symbol space \( \Sigma \). In this section we show how to do this while remaining in \( [0, 1]^2 \).

Let \((X, S)\) be a subshift of \((X, S)\) containing at least two points. For \( n \in \mathbb{N} \) we let \( B_n(Y) \) denote the set of \( r \)-blocks occurring in \( Y \) and \( \beta_r \) its cardinality. In the construction of \( \eta \) which follows, if \( b = b_1 b_2 \ldots b_r \in B_n(Y) \) and \( J = (j_1, j_2, \ldots, j_r) \in \mathbb{Z}^r \) with \( j_1 < j_2 < \ldots < j_r \), then by filling the set \( J \) with the block \( b \) we shall mean setting \( \varphi(b_{j_n}) = b_{j_n} \).

Construction. As with Oxtoby sequences, we construct \( \eta \) in steps.

Step 1. Choose \( p_r > 2 \) and set \( \eta(n) = 0 \) for \( n = -1 \mod p_r \), \( \eta(n) = 1 \) for \( n = 0 \mod p_r \).

Step 2. We let \( J(i, k) = (x_{p_r k + 1}, (k + 1) p_r - 2) \) for \( k \leq p_r - 2 \). For \( k = p_r - 1 \), we fill each \( J(i, k) \) with a different element of \( B_n(Y) \). We choose \( \beta_r > \beta_{r-1}, \beta_r > \beta_k \). For \( k + 1 \mod p_r \), \( k \leq \beta_r - 2 \) we fill \( J(i, k) \), the block that was used to fill \( J(i, \beta_r) \), so that \( \eta \) has period \( p_r \) where it is defined. (Since \( p_r > \beta_r, \eta \) remains undefined on some set \( J(k, \beta_r) \).

Step 3. Let \( J(i, k) \) be the set of \( \eta \) for \( k \leq \beta_r - 2 \) which has not been defined after the \( r \)th step, \( \eta(J(i, k)) = \eta \). For \( k = \beta_r - 1, \beta_r - 2 \) we fill each \( J(i, k) \) with a different block in \( B_n(Y) \). We then choose \( \beta_{r+1} > \beta_r, \beta_{r+1} > \beta_k \) and fill \( J(i, k) \) in the same way as \( J(i, k) \) for \( k < \beta_r \).

Since \( \beta_r > 2 \) for all \( k \) after the \( r \)th step \( \eta \) is defined on \( \mathbb{Z} \). If \( 1 \leq \beta \leq r \), then there are at least two \( r \)-blocks of \( Y \) which differ in the \( j \)-th coordinate, so the \( \beta \)-place in \( J(i, k) \) is not filled with the same element of \( [0, 1] \) for all \( k \). It follows that \( \beta_{r+1}(\eta) \) is exactly the set of integers on which \( \eta \) is defined by the end of the \( r \)th step. Thus \( \eta \) is a period structure for the Topelitz sequence \( \eta \).

Proposition 4.1. The sequence \( \eta \) is regular if and only if

\[ \sum_{i=1}^{} \frac{1}{p_r} \]

diverges.

We omit the proof, which is similar to that of Proposition 3.1. Note that for any \( 
(\mathbb{Z}) \), we can choose \( \eta \) to make this sum converge. For the rest of the section \( \eta \) will be a non-regular Topelitz sequence constructed as above. In \( \mathbb{Z} \), \( \mathcal{F} \), \( \mathcal{E}(a) \) as defined in Sect. 2 we set

\[ D = \{(x, y) \in \mathbb{Z} \mid \varphi(x) = \varphi(y) \}
\]

\[ \mathcal{G}(\mathcal{E}(a)) \times \mathbb{Z} \]

Lemma 4.2. For \( \eta \) non-regular, \( m(\mathbb{Z}(D)) = 1 \).

Proof. Since \( \mathbb{Z} \) is an \( \mathbb{Z} \)-invariant set in \( \mathbb{Z}(\mathbb{Z}) \), it must have \( m \) measure 0 or 1. For \( \eta \) non-regular, \( m(\mathbb{Z}(D)) = 1 \). so \( m(\mathbb{Z}(\mathbb{Z}(D))) = 0 \).

The same holds for \( \mathbb{Z} \).

Let \( Z = G \times Y \) with the product topology, and let \( \theta : G \to [0, 1] \) be the indicator function of \( G(\mathbb{Z}) \); that is, \( \theta(g) = 1 \) if \( g \in \mathbb{Z}(\mathbb{Z}) \) and \( \theta(0) = 0 \) otherwise. We define a map \( T : Z \to Z \) by

\[ T(g, y) = (g + 1, \theta(y)) \]

Thus \( T \) is a "piecewise power" skew product. (See Belinskaya, 1974.) \( T \) is a bimeasurable bijection. (It is not a homeomorphism.)

We define \( \psi : \sigma \to \mathbb{Z} \) by \( \psi(g, y) = \psi(y) \) to the sequence \( \sigma \) defined as follows. For \( n \in \mathbb{N} \), \( \psi(y) \) we let \( \psi(n) = \psi(y) \) where \( \psi(y) \). (This is independent of the choice of \( \alpha \).)

For \( \alpha \in \mathbb{N} \), \( \psi(y) \) we let \( \psi(n) = \psi(y) \) where \( \psi(y) \). (This is independent of the choice of \( \alpha \).) For \( \alpha \in \mathbb{N} \), \( \psi(y) \) we let \( \psi(n) = \psi(y) \) where \( \psi(y) \). (This is independent of the choice of \( \alpha \).)
(\sigma + 1)^n \text{ smallest element of } A_\text{per}(g) \cap (N \cup \{0\})$, and $\text{if } n = 0 \text{ then } \text{is the } n^{\text{th}} \text{ greatest element of } A_\text{per}(g) \cap (N \cup \{0\})$. We then set $x(n) = x(0)$. Thus we have "filled" $A_\text{per}(g)$ with as much of $y$ as will fit. If $g \in \text{End}(D)$ all of $y$ is used, so $\phi = 1 - 1$ on $x(\{D\}) \times Y$. We have positioned $y$ in $A_\text{per}(g)$ so that $y(0)$ fills the first non-negative place in $A_\text{per}(g)$. It is not hard to see that $\Sigma x = \phi(g + 1, y)$ if $0 \notin A_\text{per}(g)$ and $\Sigma x = \phi(g + 1, y)$ if $0 \in A_\text{per}(g)$; that is, $\Sigma x = \phi = T$.

**Lemma 43.** $\phi(Z) = \phi(0)$.

**Proof.** The proof is similar to that of Lemma 33. We first show $\phi(Z) = \phi(0)$. If $\alpha \in Z$, $\pi(\alpha) = 0$, then $\phi(g, y) = 0$ for all $y \in Y$. If $\alpha \notin Z$, $\pi(\alpha) = 0$, then the intervals $[-\sigma, -\sigma - 1] \times \pi(\alpha)$ increase to fill $Z$. For each $i$, $[-\sigma, -\sigma - 1] \times \pi(\alpha)$ is either empty or equal to $[-\sigma, -\sigma - 1] \times \pi(\alpha)$. The latter set is filled by an $r_i$-block of $Y$ in each $\Sigma^\infty \pi(x)$, since it is of the form $J(x, k) = \pi$ for some $k \in Z$. Hence it is filled by an $r_i$-block of $Y$ in each $\Sigma^\infty \pi(x)$. It follows that $A_\text{per}(\alpha)$ is filled by a sequence, or part of a sequence, in $Y$.

Now let $x = \phi(g, y)$ with $\pi(x) = n$, $y \in Y$. For each $n \in N$ we pick $k_n \in Z$ so that the $r_i$-block of $Y$ which fills $J(x, k_i)$ in $\eta$ matches the $r_i$-block of $Y$ which fills $[-\sigma, -\sigma - 1] \times \pi(\alpha)$ in $x$. Set $n = k_i + n_i$; then $\Sigma^\infty \eta = x$ and $x \in \phi(0)$.

**Lemma 44.** $\phi$ is bimeasurable.

We omit the proof, which is similar to that of Lemma 34.

**Theorem 45.** Let $\eta$ be a non-regular Toeplitz sequence constructed as above from $(Y, S)$. There is a one-one correspondence between the ergodic T-invariant Borel measures from $\tau$ on $Z$ and the ergodic measures on $(\phi(0), S)$ given by $\tau = \nu_1 = \phi^{-1}$.

**Proof.** If $\nu$ is a T-invariant Borel measure on $Z$ then $\nu = \phi^{-1}$ is an invariant measure on $(\phi(0), S)$. Since $\phi$ is one-to-one on $\phi^{-1}(D)$ and $\phi^{-1}(D) = \nu(D) \times Z = \nu(D) = 1$, $\nu(\pi(D) \times Z) = \nu(\phi(D) \times Z)$, so $\nu$ is ergodic. If $\nu$ is ergodic and only if $\nu$ is ergodic and only if $\nu_1 = \phi^{-1}$ is ergodic. For each $n \in N$, $\nu_1 = \phi^{-1}$ is ergodic and only if $\nu_1$ is ergodic. For each $n \in N$, $\nu_1 = \phi^{-1}$ is ergodic and only if $\nu_1$ is ergodic.

**Remark.** For each invariant Borel measure $\tau$ on $Z$, $\tau$ is a conjugacy between the measure-theoretic dynamical systems $(Z, \mathcal{M}(Z), \tau, T)$ and $(\phi(0), \mathcal{M}(\phi(0), \phi^{-1})$.

If $\lambda$ is an invariant measure on $(Y, S)$ then $m \times \lambda$ is a T-invariant Borel measure on $Z$. If $\lambda$ is ergodic, $m \times \lambda$ need not be. (See Example 4.7.) Distinct ergodic measures $\lambda, \lambda'$ on $(T, S)$ are mutually singular, so $m \times \lambda$ and $m \times \lambda'$ are mutually singular and can be decomposed into mutually singular ergodic measures. Hence $Z$ always admits at least as many ergodic measures as $Y$.

**Example 4.6.** Take $Y = \{0, 1\}^Z$. The set of ergodic measures on $(Y, S)$ has cardinality of the continuum, so the same is true for $(\phi(0), S)$.

**Remark.** For $n \in N$ we set

$$\theta(n) = \sum_{k=0}^{n-1} \theta(g + k)$$

for $g \in G$. Then

$$T^n(\phi(g, y)) = g + (g + \phi^{-1}(y))$$

for all $(g, y) \in Z$, $\phi(g, y)$ is simply the cardinality of $A_\text{per}(g) \cap (\{0\} \cup \{n\})$. If $g \in A_\text{per}(g) \cap (\{0\} \cup \{n\})$ is either $J(0, y)$ or $0$, and $\phi(g, y)$ is ergodic. We define $\phi(x, y)$ for $x \in (0, 1]^Z$.

**Example 4.7.** Let $0 < \alpha < 1$ be a cyclic permutation of $n$ points and is uniquely ergodic. We construct a Toeplitz sequence $\tau$ from $(Y, S)$, choosing $\tau_1 = \tau + 2$ so that $\tau_1 = \tau$. We claim that $(\phi(0), S)$ admits exactly $\alpha$ ergodic measures.

Let $k = 0, 1, \ldots, n - 1$ we set

$$F_k = \phi(A_k) \times (S')^n$$

and $F_0 = \phi(A_0) \cup \phi(A_1) \cup \ldots \cup \phi(A_{n-1})$. If $y = y_1 \in Y$, then $\phi(y, y_1)$ is ergodic. Hence $\phi(F_0, F_0) = F_0$, and each $F_k$ is a closed T-invariant subset of $Z$. The sets $F_k$ partition $Z$. It is not hard to see that $F_k$ supports a unique ergodic measure $\nu_k$ given by

$$\nu_k(B) = m(\phi^{-1}(B \cap F_k))$$

Thus $(\phi(0), S)$ has exactly $\alpha$ ergodic measures.
5. Calculation of Entropy

From our analysis it is easy to compute the topological entropy of each of the flows we have constructed. First, we use the well-known variational principle of Dinaburg (1970) and Goodman (1971) that the topological entropy $h(X, T)$ of a flow $(X, T)$ is the supremum over all invariant measures $v$ of the metric entropy $h_0(X, T)$. For each of our flows $\eta$, we have a map $\varphi : Z \to \Omega(u)$ (where $Z = G \times I$ for the examples of Sect. 3 and $Z = G \times Y$ for Sect. 4) which for each $T$-invariant measure $v$ is an isomorphism of the measure-theoretic dynamical systems $(Z, \varphi(Z), v, T)$ and $(\Omega(u), \theta(\Omega(u)), v = \varphi^{-1}, S)$, and so preserves metric entropy. Since $\varphi^{-1} \circ \varphi^{-1}$ is a one-to-one correspondence of invariant measures, we must have $h(Z, T) = h(\Omega(u), S)$.

Each of the flows $(Z, T)$ described in Sect. 3 is a product of the flow $(G, T)$ and the trivial flow on $I^d$. The entropy of a product is the sum of the entropies of the factors, and translation on a compact group has entropy 0. Hence the flows in Sect. 3 all have 0 entropy.

Formulas for the entropy of a piecewise power skew product appear in Belinskaya (1974) and Newton (1969). The version which is most convenient for us is in a more recent paper of Marcus and Newhouse (1979). We state a special case of the theorem to avoid making new definitions.

**Theorem 5.1.** Let $(X, R)$ and $(Y, S)$ be flows with finite topological entropy, and let $T : X \times Y \to X \times Y$ be given by $T(x, y) = (Rx, S(x, y))$, where $\theta$ is a Borel-measurable integer-valued function on $X$. Let $\pi : X \to Y$ denote the natural projection. If $v$ is invariant on $(X, R)$,

$$\operatorname{sup} h_v(X \times Y, T) = h_v(X, R) + h(X, S) \int \theta(x) d\pi$$

where the sup is taken over all $T$-invariant measures $\mu$ with $\pi(\mu) = v$.

We apply this to the flows of Sect. 4, taking $(X, R) = (G, T)$ and $Y, S, \theta$ as in Sect. 4. Then for every $T$-invariant $\mu$, $\pi(\mu) = m$, the Haar measure on $G$. Thus Theorem 5.1 reduces to

$$h(Z, T) = h_v(G, T) + h(X, S) \int \theta(x) dm\bigg|_{\pi(C)} = 0 + h(X, S) m(C) = (1 - d) h(Y, S)$$

Hence the flows of Examples 4.7 and 4.8 have entropy 0. The flow of Example 4.6 has entropy $(1 - d) \log 2$. It is possible to make $d$ arbitrarily small by choosing a rapidly increasing period structure $(p_n)$ in the construction. This yields:

**Corollary 5.2.** There exist Toeplitz $0 - 1$ sequences with entropy arbitrarily close to $\log 2$.

References