



# Robust control of I/O linearizable systems via multi-model $H_2/H_\infty$ synthesis

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## Abstract

Linearization of the input–output response of a nonlinear system via state feedback has found many applications in control of nonlinear processes. However, in the presence of uncertainties, this approach leads to inexact linearization and results in the loss of performance and stability. In this paper, a controller design methodology is developed based on input/output linearization and multi-objective  $H_2/H_\infty$  synthesis that ensures robust stability and performance. This methodology is illustrated via simulation of a regulation problem in a continuous stirred tank reactor. © 2000 Elsevier Science Ltd. All rights reserved.

*Keywords:* I/O linearization; Robust control; Nonlinear system

## 1. Introduction

In the last two decades, there has been a significant effort in the development of the theoretical foundations of the differential geometric approach to nonlinear control (Isidori, 1989; Nijmeijer & der Schaft, 1990). One of the main contributions of the differential geometric approach is Input/Output linearization (Singh & Rugh, 1972; Isidori & Ruberti, 1984), which has proved to be a very effective technique for reducing a nonlinear system to a linear one without using standard Jacobian linearization. Using the Input–Output (I/O) linearization approach, Kravaris and Chung (1987) developed the globally linearizing controller (GLC) where a nonlinear controller was designed by first using a state feedback to make the input–output relationship linear and then using an external linear controller around the input–output linear system. The GLC structure is a multi-loop configuration where an inner-loop is used for linearization and an outer-loop for stability and performance.

In principle, once the nonlinearities are canceled (or inverted), the outer-loop can be designed to impose any desired stable dynamics on the closed loop. The usual approach is to impose linear dynamics with poles in the left half plane. However, the issue of where the poles should be placed in the left half plane is not addressed in the literature. This issue is especially important when there is uncertainty in the model and the nonlinearities are not cancelled exactly. The nonlinear model will very likely represent only an approximation of the actual plant. In addition to the modeling uncertainty, there is uncertainty in the model parameters like reaction rate constants, heat transfer coefficients, etc. especially in process systems. The controller in the outer loop must be designed not only for nominal stability and performance but for robustness in face of uncertainty in the model and the environment. This robustness issue can be addressed in two alternative approaches. The first approach is to consider the effect of the uncertainty in the nonlinear model and use nonlinear techniques to account for the uncertainty. Kravaris and Palanki (1988) developed a robust nonlinear state feedback based on I/O linearization for a class of bounded perturbations to the state model using a Lyapunov function approach. Arkun and Calvet (1992) developed a robust controller methodology which guarantees the stability of the nonlinear system in

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Nomenclature			
$d_i$	disturbance vector	$x_{is}$	steady-state values
$F$	flowrate into the reactor	$z_i$	states of the linear subsystem
$k_1, k_2, k_3$	reaction rate constants	$Z_2$	the $H_2$ objective to be minimized
$L_f^r h$	$r$ th Lie derivative of $h$ along vector field $f$	$Z_\infty$	the $H_\infty$ objective to be minimized
$r$	relative degree of the system		
$u, v$	control inputs	<i>Greek letters</i>	
$V$	volume of reactor	$\Delta$	nonlinear perturbation
$W_d$	weight on the disturbances	$\theta$	set of uncertain parameters
$x_i$	states of the system	$\eta_i$	states of the zero dynamics

a neighborhood of the nominal operating point. The second approach is to consider the effect of uncertainty as a perturbation to the I/O linear model and use linear robust control techniques to account for this uncertainty. This problem of stabilizing the perturbed systems that arise in the second approach has received considerable attention recently. For instance, Christofides, Teel and Daoutidis (1996), Christofides and Daoutidis (1997) developed robust control strategies for a class of two time-scale systems where the perturbations affinely multiply the “fast” states. In this paper, we show how the second approach can be used to systematically design a state feedback for performance and robustness for I/O linearizable systems with parametric uncertainty. In particular, we use a multi-model approach to design a robust controller for the uncertain nonlinear system. In this multi-model approach, the state matrices are written as affine functions of the uncertain parameters and a controller is designed so that stability and performance specifications are met for all members in this polytopic family of models.

## 2. Problem formulation

Consider the following state–space model of a single-input–single-output (SISO) nonlinear system with parametric uncertainty

$$\dot{x} = f(x, \theta) + g(x)u, \quad y = h(x), \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state,  $u \in \mathbb{R}$  the control input,  $y \in \mathbb{R}$  the measured output, and  $\theta$  is a vector of uncertain parameters that takes values in a compact set  $\Theta \subset \mathbb{R}^p$ . For all  $\theta \in \Theta$  we assume that  $f$  and  $g$  are smooth vector fields on  $\mathbb{R}^n$ , and  $h$  is a smooth real vector valued function. The objective is to design a controller such that the closed-loop system is stable and certain performance objectives, e.g. tracking, disturbance rejection, etc., are satisfied for all  $\theta \in \Theta$ . Eq. (1) represents a large class of uncertain nonlinear chemical process. To solve this problem we propose a multi-loop design approach. The inner-loop

uses state-feedback to linearize the nominal process dynamics in the input–output sense. The outer-loop controller is a robust controller that guarantees performance despite uncertainty in the model.

In this paper we consider input–output linearizable, minimum phase nonlinear systems, with relative order  $r$ , that have well defined normal form for all  $\theta \in \Theta$ . Thus, there exists a diffeomorphism  $(\eta, z) = T(x, \theta)$  that transforms (1) into its normal form

$$\dot{\eta}_i = \phi_i(x, \theta), \quad 1 \leq i \leq n - r, \quad (2)$$

$$\dot{z}_i = z_{i+1}, \quad 1 \leq i \leq r - 1,$$

$$\dot{z}_r = \alpha(x, \theta) + \beta(x, \theta)u. \quad (3)$$

This diffeomorphism would result in a linear subsystem if the value of  $\theta$  is exactly known. However, since  $\theta$  is uncertain the diffeomorphism has to be based on some nominal value  $\theta_0$  of  $\theta$ . This results in inexact linearization which could lead to performance degradation if a conventional I/O design is used.

## 3. Robust controller design

In this section we outline a methodology for the robust controller design. We first show how system (1) is transformed by diffeomorphism based on nominal parameters using Lemma 1. The uncertain transformed system is characterized in a convenient, approximate linear form using Lemma 2. We then show a control methodology for this linear uncertain system using multi-objective  $H_2/H_\infty$  synthesis. Finally, we prove that that this controller stabilizes the original nonlinear system. We start with the following lemma.

**Lemma 1.** System (1) with an additive model for uncertainty, i.e. of the form

$$f(x, \theta) = f_0(x) + \delta f(x, \theta) \quad (4)$$

under the nominal transformation  $(\eta, z) = T(x, \theta_0)$  and the nominal feedback law  $u(x) = \beta_0(x)^{-1}[-\alpha_0(x) + v]$

renders the subsystem:

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \Delta_i, \quad 1 \leq i \leq r-1, \\ \dot{z}_r &= \Delta_r - \Delta_\beta \alpha_0 + (1 + \Delta_\beta)v, \end{aligned} \quad (5)$$

where  $\Delta_i = L_{\delta_f} L_{f_0}^{i-1}$  and  $\Delta_\beta = \beta_0^{-1} L_{\delta_g} L_{f_0}^{r-1} h$  and the subscript ‘0’ refers to the system at  $\theta = \theta_0$ .

**Proof.** For system (4) with additive model for uncertainty, the nominal transformation  $(\eta, z) = T(x, \theta_0)$  is given by

$$\begin{aligned} \eta_i &= \phi_i(x), \quad 1 \leq i \leq n-r, \\ z_i &= L_{f_0}^{i-1} h(x), \quad 1 \leq i \leq r, \end{aligned}$$

where  $\phi_i$ ,  $1 \leq i \leq n-r$  are chosen in the standard way (e.g. such that  $L_g \phi_i = 0$ ) so that  $T(x, \theta_0)$  is a diffeomorphism. In the new coordinates (1) can be written as

$$\dot{\eta}_i = \phi_i(\eta, z) + \Delta_{\eta_i}(\eta, z, \theta), \quad 1 \leq i \leq n-r, \quad (6)$$

$$\dot{z}_i = z_{i+1} + \Delta_i(\eta, z, \theta), \quad 1 \leq i \leq r-1,$$

$$\dot{z}_r = \alpha_0(x) + \Delta_r(\eta, z, \theta) + \beta_0(x)[1 + \Delta_\beta(\eta, x, \theta)]u, \quad (7)$$

where

$$\Delta_{\eta_i}(\eta, z, \theta) = L_{\delta_f} \phi_i, \quad (8)$$

$$\alpha_0(\eta, z) = L_{f_0}^r h, \quad (9)$$

$$\beta_0(\eta, z) = L_{g_0} L_{f_0}^{r-1} h. \quad (10)$$

The inner-loop controller is chosen to cancel the nominal nonlinearities as

$$u(x) = \beta_0(x)^{-1}[-\alpha_0(x) + v] \quad (11)$$

which renders Eq. (7) equal to

$$\begin{aligned} \dot{z}_i &= z_{i+1} + \Delta_i, \quad 1 \leq i \leq r-1, \\ \dot{z}_r &= \Delta_r - \Delta_\beta \alpha_0 + (1 + \Delta_\beta)v. \end{aligned} \quad (12)$$

**Remark 1.** The uncertainty in  $\theta$  induces two types of perturbations; one that acts directly on the integrators and one that acts on the control input  $v$  itself. Thus,  $v$  has to be designed for robustness with respect to both these uncertainties.

To address these robustness issues different methods have been proposed. Slotine (1984) proposed the use of sliding mode control. Alternatively, if the uncertainty is predominantly parametric, adaptive control has been proposed by Marino, Peresada and Valigi (1993). Many robust control strategies available in the literature are Lyapunov-based and are applicable only when certain “matching” conditions are satisfied. The matching conditions (Kravaris & Palanki, 1998; Chou & Wu, 1995) restrict the structure of the uncertainties in the nonlinear model. They are satisfied if there exists a smooth function

$\Delta f^*$  in  $\mathbb{R}^n$ , such that the uncertainties in Eq. (4) satisfy, 
$$\delta f(x, \theta) = g(x)\Delta f^*(x, \theta). \quad (13)$$

The matching conditions basically imply that the uncertainty in the model appears at the same order of differentiation (relative degree) as the control input. This is a rather restrictive condition. In this paper, we overcome this restrictive condition by characterizing the uncertainty in a suitable manner to design a outer loop controller as shown in Lemma 2.

Differential geometric techniques such as input–output linearization use coordinate transformation and state feedback to reduce the nonlinear system to a linear one. However, in the presence of uncertainties, these methods do not give perfectly linear models. Perturbations appear in the canonical form, as nonlinear functions of  $z$ , due to the presence of uncertainties. For the use of linear robust control techniques these nonlinearities have to be linearized. Standard Jacobi linearization of these nonlinear perturbations around the steady states can be used for this purpose. We note that this is different from the Jacobi linearization of the original nonlinear system. Only the perturbations arising due to uncertainties are linearized but not the whole model.

**Lemma 2.** The system of form (5) can be characterized as

$$\dot{z} = A(\theta)z + B(1 + W_v \Delta_v)v + W_d d, \quad y = Cz, \quad (14)$$

where  $\|d_i\|_2 \leq 1$  are the nonlinear perturbations represented as external bounded disturbances and  $W_v \Delta_v$  is a linear multiplicative representation of the uncertainty in the input  $\Delta_\beta$ .  $W_d$  and  $W_v$  are linear time invariant weights.

**Proof.** By formal Taylor series expansion we can write

$$\Delta_i = \delta_i(\theta)z + \tilde{\delta}_i(\eta, z, \theta), \quad 1 \leq i \leq r-1, \quad (15)$$

$$\Delta_r - \Delta_\beta \alpha_0 = \delta_r(\theta)z + \tilde{\delta}_r(\eta, z, \theta), \quad (16)$$

where  $\delta_i(\theta)$ ,  $1 \leq i \leq r$  are row vectors arising from the Taylor series expansion and  $\tilde{\delta}_i$  and  $\tilde{\delta}_r$  contain the higher-order terms. Then the system becomes

$$\dot{z} = A(\theta)z + \Delta_A + B(1 + \Delta_\beta)v, \quad y = Cz, \quad (17)$$

where  $B = (0, \dots, 0, 1)^T$ ,  $C = (1, 0, \dots, 0)$  and  $\Delta_A = (\tilde{\delta}_1, \dots, \tilde{\delta}_r)^T$ . The nonlinear perturbations  $\Delta_A$  are represented as external bounded disturbances. Let  $d_i \in \mathcal{L}_2[0, \infty)$ , such that  $\|d_i\|_2 \leq 1$ ,  $1 \leq i \leq r$ . Stable linear time invariant weights  $W_d$  are chosen such that

$$\|\tilde{\delta}_i\|_2 \leq \|W_d d_i\|_2, \quad 1 \leq i \leq r. \quad (18)$$

Then the effects of  $\Delta_A$  can be represented by  $W_d d$ , where  $W_d = \text{diag}(W_{d_1}, \dots, W_{d_r})$ . The uncertainty in the input  $\Delta_\beta$  is represented as a linear multiplicative uncertainty,  $W_v \Delta_v$ , such that

$$\sup_{\theta \in \Theta} \|\Delta_\beta\| \leq \|W_v \Delta_v\|_\infty, \quad (19)$$

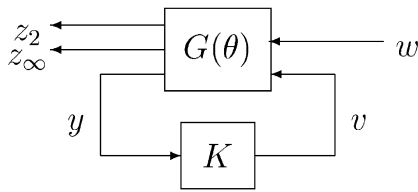


Fig. 1. Multi-model  $H_2/H_\infty$  synthesis problem.

where  $\|A_v\|_\infty < 1$  and  $W_v$  is a stable linear time invariant weight. This reduces Eq. (17) to

$$\dot{z} = A(\theta)z + B(1 + W_v \Delta_v)v + W_d d, \quad y = Cz. \quad (20)$$

To complete the design we must find a robustly stabilizing controller for the uncertain system (14). The  $H_\infty$  objective in the robust controller design is to cancel the effect of worst-case disturbances (the nonlinear perturbations) and the  $H_2$  objective is to obtain the optimal

LQG control. This linear robust control problem can be solved via multi-objective optimization techniques such as mixed  $H_2/H_\infty$  synthesis with pole placement constraints. This technique can be used for robust design when the linear fractional representation of the plant is affine in  $\theta$ . The multi-model  $H_2/H_\infty$  state-feedback synthesis places the poles such that the system has good performance for all values of  $\theta$ . This problem is represented in Fig. 1.  $w$  contains all external disturbances, e.g.  $d$ , and  $Z_2$  and  $Z_\infty$  contains the relevant errors signals that we want to maintain small with respect to the 2-norm (average) and  $\infty$ -norm (worst case), respectively. The generalized plant  $G(\theta)$  represents the plant model together with performance and normalization weights and is affine in  $\theta$ .

The objective is to find a stabilizing controller  $K$  such that

$$a\|T_{Z_\infty w}\| + b\|T_{Z_2 w}\| \quad (21)$$

is minimized, for all  $\theta \in \Theta$ , where  $T_{Z_\infty w}$  and  $T_{Z_2 w}$  are linear operators mapping  $w$  to  $Z_\infty$  and  $w$  to  $Z_2$ , respectively, and  $a, b$  are positive numbers representing the trade-off between the  $H_2/H_\infty$  objectives.

The multi-objective synthesis problem for an uncertain state-space realization can be solved using linear matrix inequalities (LMI). First, the uncertain state-space model (17) is represented as a polytopic family of systems where the state-space matrices are affine functions of the uncertain parameters, i.e. of the form

$$A(\theta) = A_0 + \theta_1 A_1 + \dots + \theta_k A_k + \dots + \theta_p A_p, \quad (22)$$

where,  $p$  is number of uncertain parameters. Then, multi-objective problem (21) is solved by LMI using the following theorem.

**Theorem 1.** (Khargonekar & Rotea, 1991). *Given a polytopic family of LTI systems, of the form*

$$\dot{z} = A(\theta)z + B_1(\theta)d + B_2(\theta)v, \quad (23)$$

$$Z_\infty = C_1 z + D_{11}d + D_{12}v, \quad (24)$$

$$Z_2 = C_2 z + D_{22}v. \quad (25)$$

The state feedback  $v = Kz$  that robustly stabilizes the above system and minimizes the performance objective is given by  $K = YX^{-1}$ , where  $X$  and  $Y$  are obtained by solving the following LMI formulation of the multi-objective state feedback synthesis problem:

Minimize  $a\gamma^2 + b \text{Trace}(Q)$  over  $Y, X, Q$  and  $\gamma^2$  satisfying

$$\begin{pmatrix} A_k X + X A_k^T + B_{2k} Y + Y^T B_{2k}^T & B_{1k} & X C_1^T + Y^T D_{12}^T \\ B_{1k}^T & -I & D_{11}^T \\ C_1 X + D_{12} Y & D_{11} & -\gamma^2 I \end{pmatrix} < 0, \quad (26)$$

$$\begin{pmatrix} Q & C_2 X + D_{22} Y \\ X C_2^T + Y^T D_{22}^T & X \end{pmatrix} > 0, \quad (27)$$

$$\text{Trace}(Q) < v_0^2, \quad (28)$$

$$\gamma < \gamma_0^2, \quad (29)$$

$$f_\Theta < 0, \quad (30)$$

where  $A_k, B_{1k}, B_{2k}$  are coefficients in the polytopic representation (as shown in Eq. (22)) of the parameter dependent state matrices  $A, B_1, B_2$ , respectively,  $\gamma$  and  $v$  are upper bounds on the  $H_\infty$  and  $H_2$  norms, respectively, and  $f_\Theta$  specifies the pole placement constraints.

The minimization problem posed by Theorem 1 can be solved using software such as the LMI control toolbox in MATLAB (Gahinet, Nemirovski, Laub & Chilali, 1995). We now show that the feedback controller found by multi-objective synthesis robustly stabilizes the original nonlinear system.

**Theorem 2.** Consider a system of the form of Eq. (14) and assume that

1.  $\theta$  is in a compact set,
2.  $\|A_v\|_\infty < 1$ ,
3.  $\|d_i\|_2 \leq 1$ ,
4.  $W_d$  and  $W_v \in \mathbb{R}H_\infty$ .

If a controller  $K$  robustly stabilizes the system represented by Eq. (14), then this controller also robustly stabilizes the nonlinear system represented by Eq. (1).

**Proof.** According to Lemma 1, characterization of the nonlinear perturbations that appear in Eq. (12), introduces the linear part of the Jacobi linearized perturbations into the state matrix,  $A$  and higher order terms as disturbances. If one can find a linear robust controller  $K$  that stabilizes this uncertain system in presence of the disturbances, then it stabilizes subsystem (5). To see this consider a Lyapunov function for the linear system (14).

$$V = z^T z. \quad (31)$$

If  $v = -Kz$  is chosen such that system (14) is robustly stable, then

$$\begin{aligned} \dot{V} &= z^T(A(\theta) + A^T(\theta))z + 2z^TB(1 + W_v\Delta_v)v \\ &\quad + 2z^TW_d d \leq 0. \end{aligned} \quad (32)$$

The nonlinear system (5) can be written as

$$\dot{z} = (F + \delta(\theta))z + B(1 + W_v\Delta_v)v + \tilde{\delta}(\eta, z, \theta), \quad (33)$$

where

$$F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \quad (34)$$

$\tilde{\delta}(\eta, z, \theta) = (\tilde{\delta}_1, \dots, \tilde{\delta}_r)^T$ ,  $\delta(\theta) = (\delta_1, \dots, \delta_r)^T$  and  $B = (0 \ 0 \dots 1)^T$ . Considering the same type of Lyapunov function for this system, we have

$$V_n = z^T z \quad (35)$$

and

$$\begin{aligned} \dot{V}_n &= z^T(F + \delta(\theta) + F^T + \delta^T(\theta))z \\ &\quad + 2z^TB(1 + W_v\Delta_v)v + 2z^T\tilde{\delta}(\eta, z, \theta). \end{aligned} \quad (36)$$

Note that the nonlinearities are characterized as disturbances  $d$  according to Eq. (18) and  $A(\theta)$  captures the linear part of Eq. (33). Since the disturbances are characterized according to Eq. (18), the last term in Eq. (36) is always less than the last term in Eq. (32). Hence, it follows that

$$\dot{V}_n \leq 0. \quad (37)$$

The controller  $K$ , therefore stabilizes Eq. (5). Since  $\theta$  is in a compact set the zero dynamics (6) are stable for all values of  $\theta$ . This ensures that  $K$  stabilizes the original system (1). The above results ensure global stability if the zero dynamics are globally stable for all  $\theta \in \Theta$ .

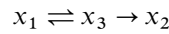
**Remark 2.** Since performance objectives can be characterized as robust stabilization (Doyle, 1982) the above theorem is not limited to stabilization alone.

**Remark 3.** Note that if a linear controller  $K$  cannot be found by solving the optimization problem (21) in

Theorem 1, this does not imply that a robustly stabilizing controller does not exist for the original uncertain nonlinear system. This situation can arise when a bound on  $\tilde{\delta}$  cannot be established or when the bound on  $\tilde{\delta}$  is so large that the performance level  $\gamma$  cannot be satisfied for the uncertainty.

#### 4. Illustrative example

Consider a continuous stirred-tank reactor (CSTR) in which an isothermal, liquid phase, multi-component chemical reaction is being carried out. The chemical reaction system is



with the rates of reaction given by

$$r_1 = k_1 x_1 - k_2 x_3^2, \quad (38)$$

$$r_2 = k_3 x_3^2. \quad (39)$$

The objective is to keep  $x_2$  at a desired set-point concentration by manipulating the molar feedrate of species  $x_3$ . This system can be modeled as (Kravaris & Palanki, 1988)

$$\dot{x}_1 = -k_1 x_1 + \frac{F}{V}(C_{AF} - x_1) + k_2 x_3^2,$$

$$\dot{x}_2 = -\frac{F}{V}x_2 + k_3 x_3^2,$$

$$\dot{x}_3 = k_1 x_1 - \frac{F}{V}x_3 - (k_2 + k_3)x_3^2 + u,$$

$$y = x_2.$$

There is uncertainty in the parameter  $k_2$  which is represented as  $k_2 = \hat{k}_2 + c_2\theta$ ,  $|\theta| < 1$ , where  $\hat{k}_2$  is the nominal value of  $k_2$  and  $c_2 = 2.9$  is a scaling constant representing the magnitude of the uncertainty. The system parameters are:  $k_1 = 1$ ,  $\hat{k}_2 = 3$ ,  $k_3 = 5$ ,  $F = 3$  and  $V = 3$ . This system has well defined relative degree  $r = 2$  for all values of  $k_2$ . At the desired operating steady state,  $x_{1s} = 2.1799$ ,  $x_{2s} = 3.933$ ,  $x_{3s} = 0.869$ , and  $u_s = 5$ . Note that this system does not satisfy the matching conditions (13) and hence Lyapunov methods proposed in literature cannot be used. Using Lemmas 1 and 2, we obtain the quasi-linear subsystem

$$\dot{z}_1 = z_2, \quad (40)$$

$$\dot{z}_2 = -3\frac{F}{V}c_2 x_{3s}\theta z_1 - 3c_2 x_{3s}z_2 + W_d d + v, \quad (41)$$

where  $d$  is an external disturbance representing the effects of  $\tilde{\delta}$ , and  $W_d$  captures the size and frequency spectrum of the perturbations. Note that the state matrix is an affine in  $\theta$ .

If we ignore the uncertainty, then we can use  $v$  to impose any desired linear dynamics. We choose to place the poles at  $-1 \pm j$  for good transient response with optimal damping. Thus, a standard I/O design will give

$$v = -2z_1 - 2z_2. \tag{42}$$

Now, we design a state feedback control  $v = Kz$  for disturbance rejection and robustness using a multi-model  $H_2/H_\infty$  synthesis with pole placement constraints. From simulations we found that  $W_d = 10$ . The  $H_\infty$  objective is to minimize the effect of the worst case disturbance  $d$  on the output  $y$ . The  $H_2$  objective is to minimize the 2-norm of the operator from  $d$  to  $(z_1, z_2, v)^T$ . The pole placement constraints are chosen compatible with the previous design. To limit the bandwidth poles are constrained within a semi-circle of radius  $\omega_n = 80$  rad/s. The linear subsystem can be written as

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -3c_2x_{3s}F\theta & -3c_2x_{3s}\theta \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ W_d \end{bmatrix} d + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v, \\ y &= z_1. \end{aligned} \tag{43}$$

The state matrix can be written as

$$A(\theta) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3c_2x_{3s} \end{bmatrix} \theta \begin{bmatrix} F/V & 1 \end{bmatrix}. \tag{44}$$

The  $H_2$  and  $H_\infty$  objectives can be written as

$$Z_\infty = z_1, \tag{45}$$

$$Z_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v. \tag{46}$$

Using Theorem 1 and the LMI control toolbox (Gahinet et al., 1995) we obtain

$$v = -83z_1 - 73z_2. \tag{47}$$

Simulations were carried out for the nonlinear system with both controllers for several values of  $k_2$ . These results are plotted in Figs. 2 and 3 which show the deviation of the output from the desired steady state with time. It is observed that the conventional I/O controller is unable to keep the system at the desired trajectory when  $k_2$  is uncertain. However, the robust I/O design is able to account of this uncertainty.

Simulations were also conducted to compare the performance of this controller with a standard PI controller. The PI controller was designed by linearizing the nominal nonlinear system around the nominal steady state and choosing the controller parameters  $k_c$  and  $\tau_I$  via simulations. The results are shown in Fig. 4. It is observed that while the PI controller is able to regulate the

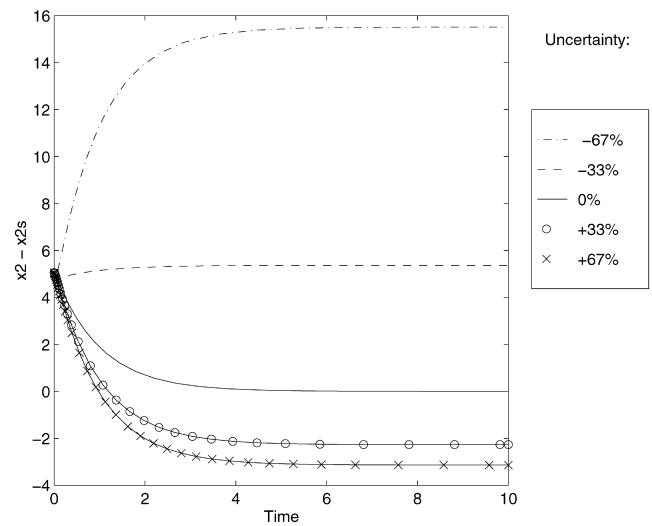


Fig. 2. Conventional I/O controller ( $x_0 = [1.0 \ 8.33 \ 1.0]$ ).

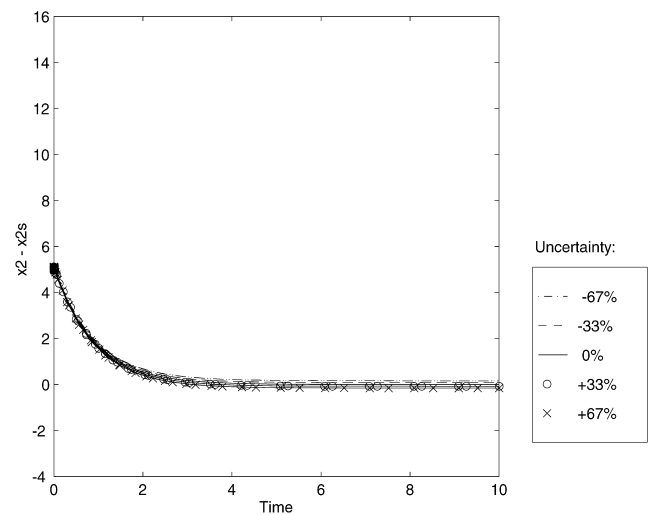


Fig. 3. Robust I/O controller ( $x_0 = [1.0 \ 8.33 \ 1.0]$ ).

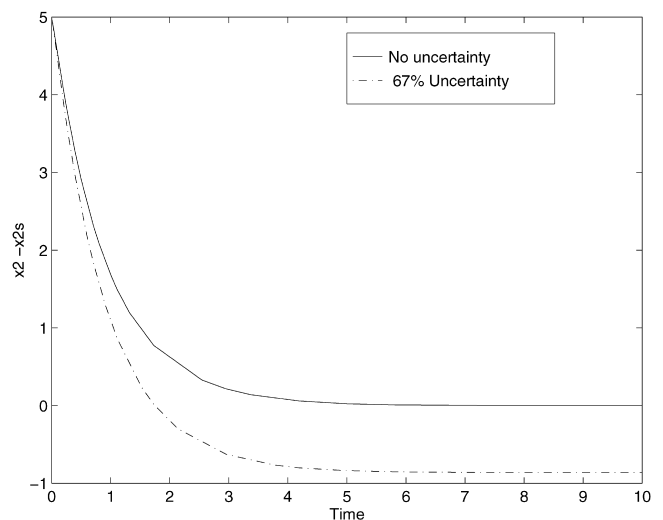


Fig. 4. PI controller ( $K_c = 0.4 \ \tau_I = 2.67$ ).

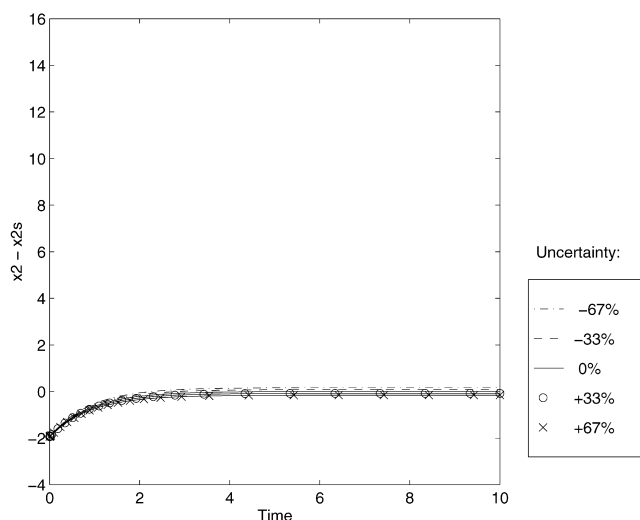


Fig. 5. Robust I/O controller ( $x_0 = [4.0 \ 1.33 \ 6.0]$ ).

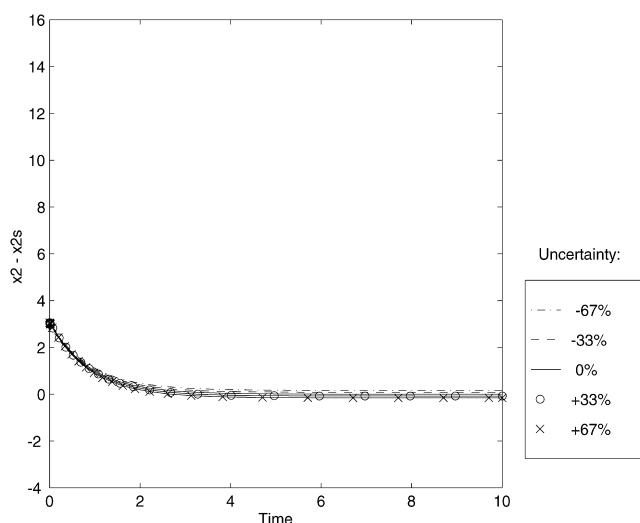


Fig. 6. Robust I/O controller ( $x_0 = [4.0 \ 6.33 \ 1.0]$ ).

nominal system to the desired steady state, it performs poorly in the face of parametric uncertainty. While it may be possible to “detune” the PI parameters to improve robustness there is no systematic procedure to guarantee the desired performance.

The efficacy of the robust nonlinear controller to regulate the system starting from different initial conditions was also studied. The results are shown in Figs. 5 and 6. It is seen that the controller performs satisfactorily for a wide range of initial conditions.

## 5. Conclusions

A design procedure was developed for a class of uncertain nonlinear systems based on I/O linearization and multi-objective  $H_2/H_\infty$  synthesis. This procedure combines the advantages of I/O linearization and linear robust control techniques to guarantee performance for a class of nonlinear systems with uncertainty. This approach does not require restrictive matching conditions to be satisfied. These controllers can be designed using off the shelf software. This methodology is illustrated via simulation of a regulation problem in a CSTR.

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