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# Nonlinear control of nonsquare multivariable systems

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**Abstract**

This paper is concerned with the synthesis of a nonlinear state feedback law for nonsquare multivariable nonlinear systems. Previous approaches in the literature have solved this problem by (1) squaring the system by discarding some inputs or by adding new outputs, or (2) by utilizing some inputs for input/output (I/O) linearization and the remaining inputs for minimizing cost. In this paper, a nonlinear feedback law is synthesized which utilizes all the available inputs to I/O linearize the system and minimize the cost of the control effort by solving a convex optimization problem on-line. This procedure is illustrated via simulation of a regulation problem in a nonlinear continuous stirred tank reactor with three inputs and two outputs. 

*Keywords:* Input–output linearization; Nonsquare systems; Pseudo-inverse

## 1. Introduction

Processes having unequal number of inputs and outputs occur frequently in the chemical process industry. Such systems are called nonsquare systems. However, for controller design purposes, they are often "squared" by adding or deleting the appropriate number of inputs or outputs from the system matrix (Reeves & Arkun, 1989). Once such systems are squared, then controller synthesis procedures developed for square systems (where the number of inputs and outputs is equal) are applied directly. However, it has been shown that there can be advantages in synthesizing a controller for the original nonsquare system. For instance, Morari, Grimm, Ogelsby, and Prosser (1985) compared square and nonsquare structures of a reactor application study and concluded that for their system the nonsquare structure was less sensitive to modeling errors due to a smaller condition number. Despite such evidence, the literature on controller design is sparse. Most of the available results in the literature of nonsquare systems are for linear systems. For instance, Treiber (1984) demonstrated the application of Rosenbrock's direct nyquest array method to design a multivariable control scheme for a ternary distillation column modeled with nonsquare transfer functions. A precompensator analogous to the inverse gain array for square systems was used to square and decouple the system. Treiber and Hoffman (1986) utilized a similar approach for the control of a vacuum distillation column with five inputs and four outputs. Lau, Alvarez, and Jensen (1985) utilized a singular value decomposition (SVD) framework to synthesize a control strategy for nonsquare linear systems. Reeves & Arkun (1989) derived a block relative gain array measure for decentralized control structure design of nonsquare linear systems. Chang and Yu (1990) extended Bristol's relative gain array (RGA) to nonsquare linear systems and used this measure to assess performance based on steady-state information.

However, these results are not applicable directly to nonsquare *nonlinear* systems. Nonlinearities appear in almost all process control systems and this provides the motivation for the development of a nonlinear controller synthesis framework for nonsquare nonlinear systems. In the past two decades, differential geometric methods have been used to develop a controller synthesis framework for *square* nonlinear systems (Brockett, 1989; Isidori, 1989). Some of the important results in the area are the solutions of problems of reachability (Lobry, 1970), input/state linearization (Jakubcsyk & Respondek, 1982; Su, 1982; Hunt, Su, & Meyer, 1983) and

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input/output (I/O) linearization (Singh & Rugh, 1972; Claude, Fliess, & Isidori, 1983; Isidori & Ruberti, 1984).

I/O linearization is a multi-loop configuration in which an inner loop uses nonlinear state feedback laws so that the input–output behavior is exactly linear and the outer loop is used for stability and performance. Kravaris and Soroush (1990) utilized the I/O linearization framework for multi-input–multi-output (MIMO) systems in conjunction with an external multivariable controller with integral action to derive a globally linearizing controller (GLC). Daoutidis and Kravaris (1991) studied inversion and zero dynamics of multivariable systems in the GLC framework.

However, all the above results have been derived for "square" systems. Very few results are available for controller design of nonsquare systems in the I/O framework. Doyle, Allgower, Oliveria, Gilles, and Morari (1992) and McClamroch and Schumacher (1993), investigated the design of I/O linearizing controllers for non-minimum-phase nonlinear systems with two inputs and one output. The first input was used to achieve I/O linearization while the second input was used to stabilize the otherwise unstable zero dynamics. For nonsquare systems with more inputs than outputs and equal relative degree with respect to all inputs McLain, Kurtz, Henson, and Doyle (1996) derived an analytical expression, where some inputs were used for I/O linearization and the remaining inputs were used to minimize input cost. When the relative degrees are different, McLain et al. (1996) suggested that the inputs be redefined via dynamic extension to make the relative degrees equal. Another possible approach to handle nonsquare multi-input/multi-output (MIMO) control problems is to "square" the system by adding new inputs or discarding some outputs, as proposed in the linear case (Reeves & Arkun, 1989; Le & Safonov, 1992). However, in nonlinear systems, there is no systematic procedure for introducing additional outputs or discarding existing inputs. By discarding existing inputs one may be giving up flexibility in controller design. Furthermore, the introduction of new outputs might require additional control effort.

In this paper, we propose an alternative strategy, based on I/O linearization that can be applied directly to nonsquare MIMO systems without having to use a "squaring" procedure. This procedure minimizes control effort, by solving an optimization problem on-line.

The reminder of the paper is organized as follows. In Section 2, the nonsquare MIMO problem is formulated. In Section 3, a nonlinear controller design procedure for handling nonsquare MIMO systems is presented. Connections between this procedure and the calculation of a Moore–Penrose pseudo-inverse are presented and zero dynamics of nonsquare systems are defined. This procedure is illustrated via simulation of a continuous stirred tank reactor (CSTR) example in Section 4. Finally, Section 5 contains a summary and conclusions.

## 2. Problem formulation

Consider the following state-space realization of a MIMO nonlinear system with $n$ states, $m$ inputs and $p$ outputs, $p \leqslant m \leqslant n$.

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x)u_j, \qquad y_i = h_i(x), \quad i = 1, \ldots, p, \tag{1}$$

where

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix} \in \mathbb{R}^p, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

are the inputs, outputs and states, respectively. $f(x)$ is a smooth vector field on $\mathbb{R}^n$, $g_1(x), \ldots, g_m(x)$ are smooth vector fields on $\mathbb{R}^m$ and $h_1(x), \ldots, h_p(x)$ are smooth scalar fields. The cost associated with the inputs is represented as follows:

$$J = w_1^2 u_1^2 + w_2^2 u_2^2 + \cdots + w_m^2 u_m^2 = \|Wu\|_2^2, \tag{2}$$

where, $w_i$ is the cost of $u_i$, $W$ is a diagonal matrix with $w_i$ as its elements and $J$ represents the total cost of the inputs.

In this formulation, it is assumed that the number of inputs is greater than or equal to the number of outputs. This is a fairly typical situation in the fine chemicals and pharmaceutical industry where several inputs (such as feedrate, pH, temperature) are available to control one output (such as product concentration). The case where there are more outputs than inputs typically leads to an uncontrollable situation and is not considered here.

Most of the results available in the literature consider *square* MIMO systems. In order to describe the regulation problem precisely, we define the characteristic matrix specifically for *nonsquare* MIMO systems. This definition for nonsquare systems is analogous to the definition for square systems in Isidori (1989). We first review the definition of relative degree for nonsquare systems.

**Definition 1** (Isidori, 1989). A nonlinear MIMO system of the form (1) is said to have a relative degree $r_i$ with respect to an output $y_i$ if the vector

$$\begin{aligned} &L_g L_f^k h_i(x) \triangleq [L_{g_1} L_f^k h_i(x) \ldots L_{g_m} L_f^k h_i(x)] = \bar{\mathbf{0}}, \\ &\quad k = 0, 1, \ldots, r_i - 2, \\ &L_g L_f^k h_i(x) \triangleq [L_{g_1} L_f^k h_i(x) \ldots L_{g_m} L_f^k h_i(x)] \neq \bar{\mathbf{0}}, \\ &\quad k = r_i - 1. \end{aligned} \tag{3}$$

Essentially $r_i$ is the smallest integer $k$ for which the vector $L_g L_f^{k-1} h_i(x)$ has at least one nonzero component. This means that at least one of the inputs $u_j$ affects the output $y_i$ after $r_i$ integrations.

**Definition 2.** If system (1) has a well-defined relative degree $r_i$ for each output $y_i$ then the characteristic matrix of the system is the following $p \times m$ matrix:

$$\beta(x) = L_g L_f^{r-1} h(x) = \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x) & \dots & L_{g_m} L_f^{r_1-1} h_1(x) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_p-1} h_p(x) & \dots & L_{g_m} L_f^{r_p-1} h_p(x) \end{bmatrix}_{p \times m}. \quad (4)$$

If a system represented by (1) has well-defined relative degree $r_i$ for all outputs $y_i$ with $r = \sum r_i$ $(r \leqslant n)$ and the characteristic matrix (4) has full row rank ($p$), then there exists a diffeomorphism $(\eta,\xi) = T(x)$ given by

$$\xi^i = \begin{bmatrix} \xi_1^i \\ \xi_2^i \\ \vdots \\ \xi_{r_i}^i \end{bmatrix} = \begin{bmatrix} h_i(x) \\ L_f h_i(x) \\ \vdots \\ L_f^{r_i-1} h_i(x) \end{bmatrix}, \quad i = 1, \dots, p, \quad (5)$$

$$\eta_i = \phi_{r+i}(x), \quad i = 1, \dots, n-r, \quad (6)$$

where $d\phi_{r+1}, d\phi_{r+2}, \dots, d\phi_n$ are chosen such that

$$[d\xi_1^1,\ d\xi_2^1, \dots,\ d\xi_{r_1}^1, \dots,\ d\xi_1^p,\ d\xi_2^p, \dots,\ d\xi_{r_p}^p, \dots,\ \phi_{r+1},\ \phi_{r+2}, \dots, \phi_n]$$

are linearly independent and $\xi = [\xi^1,\ \xi^2, \dots, \xi^p]$, $\eta = [\eta_1,\ \eta_2, \dots,\ \eta_{n-r}]$. This transforms system (1) to the normal form:

$$\dot{\eta} = q(\xi,\eta) + \sum_{j=1}^{m} p_j(\xi,\eta) u_i, \quad (7)$$

$$\begin{aligned} \dot{\xi}_1^i &= \xi_2^i, \\ &\vdots \\ \dot{\xi}_{r_i-1}^i &= \xi_{r_i}^i, \quad (8) \\ \dot{\xi}_{r_i}^i &= \alpha_i(\xi,\eta) + \sum_{j=1}^{m} \beta_{ij}(\xi,\eta) u_j, \quad i = 1, \dots, p, \end{aligned}$$

where $\alpha_i = L_f^{r_i} h_i(x)$ and $\beta_{ij}$ is the $(i,j)$th entry in (4).

Eqs. (8) give the structure of $p$ subsystems, which form the linearizable subsystem.

Once the system has been transformed to the above normal form, feedback laws can be designed to cancel the nonlinearities that appear in the equations

$$\dot{\xi}_r = \alpha(\xi,\eta) + \beta(\xi,\eta) u, \quad (9)$$

where

$$\dot{\xi}_r = \begin{bmatrix} \dot{\xi}_{r_1}^1 \\ \dot{\xi}_{r_2}^2 \\ \vdots \\ \dot{\xi}_{r_p}^p \end{bmatrix}, \quad \alpha = \begin{bmatrix} L_f^{r_1} h_1(x) \\ L_f^{r_2} h_2(x) \\ \vdots \\ L_f^{r_p} h_p(x) \end{bmatrix},$$

and $\beta$ is the characteristic matrix (4) and $u$ is the input vector.

The underlying principle of input–output linearization is to define new reference inputs and use a feedback law that renders the sub-system linear in the I/O sense. Then, a controller is designed for this linear sub-system using standard linear control techniques and the actual inputs are found from the values of the new reference inputs.

When the MIMO system under consideration is square ($m = p$) and the cost (2) of the inputs is ignored, I/O linearization can be achieved by defining a vector of new inputs, $v$, and using the feedback law:

$$\Phi(\eta,\xi,v) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} = \beta_{m \times m}^{-1} \left[ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} - \begin{bmatrix} \alpha_1(\xi,\eta) \\ \alpha_2(\xi,\eta) \\ \vdots \\ \alpha_m(\xi,\eta) \end{bmatrix} \right] \quad (10)$$

provided $\beta$ is invertible. The problem addressed in this paper is to find a similar feedback law for nonsquare systems, which makes the input–output behavior of the nonsquare system linear.

## 3. I/O linearization for nonsquare systems

For a nonsquare MIMO system represented by (1), $\beta$ is a $p \times m$ matrix; $\beta^{-1}$ is not defined. Thus, Eq. (10) cannot be used to define new inputs required for I/O linearization.

We propose to use all the available $m$ inputs in the original system. However, we define $p$ new external reference inputs to linearize the system. This implies that we use a feedback law such that the linear subsystem is square and each new input $v_i$ $(1 \leqslant i \leqslant p)$ is a blend of available inputs, $u_j$ $(1 \leqslant j \leqslant m)$. In most practical situations, there is a cost associated with each input. If there is a choice in selecting inputs, one would try to minimize the total cost of inputs. The presence of additional inputs introduces a degree of flexibility. These factors can be utilized to obtain the "best" choice of inputs. This structure allows for the use of all available inputs and also conserve the control effort as the number of outputs is not changed.

The construction of the I/O linearizing feedback law, for nonsquare systems requires solution to the following optimization problem. Consider the solution to a set of algebraic equations in which the number unknowns is greater than the number of equations, i.e. of the form

$$Ax = b. \quad (11)$$

There are infinite solutions to the above set of equations. The solution that minimizes a cost function defined by $J = \|Wx\|_2^2$, is unique and can be obtained by solving the following problem:

$$\begin{aligned} \min \quad & \|Wx\|_2^2 \\ \text{s.t.} \quad & Ax = b. \end{aligned} \tag{12}$$

The solution to the above optimization problem can be interpreted as finding a pseudo-inverse of $A$. In other words, the solution that is obtained can be written as $x = A^{\#}b$, where $A^{\#}$ is the Moore–Penrose pseudo-inverse of $A$ (Noble, 1976).

**Remark 1.** The existence of the pseudo-inverse of the characteristic matrix is guaranteed by the assumption that it has full row rank. This is analogous to the assumption in the square MIMO systems that the characteristic matrix is invertible.

**Theorem 1.** *Consider a nonsquare MIMO system in the normal form given by Eqs.* (7) *and* (8). *In the presence of input weights and constraints*, the I/O *linearizing feedback law for this system is*

$$u = \beta^{\#}(v - \alpha), \tag{13}$$

*where $\beta^{\#}$ is a Moore-Penrose pseudo-inverse that accounts for the costs* (2) *of the inputs and $v$ is a vector of $p$ external reference inputs. This induces the linear behavior given by*

$$\begin{aligned} y_i &= \xi_1^i = \xi_2^i, \\ \frac{\mathrm{d}y_i}{\mathrm{d}t} &= \dot{\xi}_2^i = \xi_3^i, \\ &\vdots \\ \frac{\mathrm{d}^{r_i}y_i}{\mathrm{d}t^{r_i}} &= \dot{\xi}_{r_i}^i = v_i, \quad i = 1, \ldots, p. \end{aligned} \tag{14}$$

**Proof.** This theorem is proved by construction. Suppose that we choose a control law

$$u = \Phi(\eta, \xi, v) \tag{15}$$

such that the resulting structure is (14).

Now standard linear multivariable control techniques (Skogestad & Postlethwaite, 1996) can be used to design a controller for this linear subsystem. Once the controller is designed, i.e., $v_i$ are known as functions of $\xi_j^i$, we can find the linearizing feedback law $u = \Phi(\eta, \xi, v)$.

We are seeking $u = [u_1 \;\; u_2 \ldots u_m]^{\mathrm{T}}$ that is a solution to the system of equations

$$\alpha_i(\xi, \eta) + \beta_i(\xi, \eta)u = v_i, \quad i = 1, \ldots, p, \tag{16}$$

where $\beta_i(\xi, \eta)$ is the $i$th row of the characteristic matrix. Since $u$ is a vector of dimension $m$ and $v$ is a vector of dimension $p$, the number of unknowns ($m$) is greater that the number of equations ($p$). A unique solution is found by considering the following optimization problem, at each point, which minimizes the cost of inputs:

$$\begin{aligned} \min \quad & \|Wu\|_2^2 \\ \text{s.t.} \quad & \alpha + \beta u = v, \end{aligned} \tag{17}$$

where, $W$ is a diagonal matrix of input weights which reflect their cost. This convex optimization problem can be solved numerically, using standard optimization techniques such as sequential quadratic programming (SQP) (Edgar & Himmelblau, 1989).

The solution of the optimization problem posed by Eq. (17) can be interpreted as calculating pseudo-inverse $\beta^{\#}$. Hence, the I/O linearizing feedback law for nonsquare systems is given by Eq. (13). □

**Remark 2.** Note that the squaring down procedure will remove $m - p$ inputs from the set of Eqs. (16) and the squaring up procedure adds $m - p$ outputs (so, $m - p$ new reference inputs $v$). Thus, both these procedures lead to a loss in design flexibility. Furthermore, as shown in the illustrative example, the squaring down procedure may lead to increased cost. For this reason, it preferable to utilize all the available inputs $u_1, u_2, \ldots, u_m$ to control the outputs $y_1, y_2, \ldots, y_p$.

**Remark 3.** The resulting linear subsystem is decoupled with respect to the new inputs $v$, i.e., the $i$th output is affected only by the $i$th new input, which is a blend of the original inputs.

**Remark 4.** In the case of the nonsquare MIMO systems with equal relative degrees and no input costs, the solution to the optimization problem (17) is equivalent to the solution provided by McLain et al. (1996). When the system is square with no input costs, the feedback law given by Eq. (13) coincides with the feedback law given by Kravaris and Soroush (1990) for square MIMO systems since the Moore–Penrose pseudo-inverse for a square system is the same as the inverse of the system.

### *3.1. Procedure for I/O linearizing control of nonsquare systems*

The procedure to solve the nonsquare MIMO problem is as follows.

(1) Calculate the relative degree $r_i$ for each of the $p$ outputs.
(2) Transform the original system to the normal form (7) and (8), using the diffeomorphism (5) and (6).
(3) Introduce $p$ new reference inputs to cancel the nonlinearities in subsystem (8).
(4) Use linear multivariable control techniques to design a controller for the resulting linear-subsystem (14) in terms of the external reference inputs $v_i$.
(5) At each time step, solve the optimization problem (17) numerically.

(6) Use the optimal input solution in the linearizing inner-loop controller.

Note that in this procedure, the external linear loop is designed *first* and the I/O linearizing inner loop is designed *later*. The optimization problem in Step 5 for calculating the pseudo-inverse is solved on-line. This on-line problem requires the minimization of a *quadratic* objective function subject to *linear* equality constraints of the form of Eq. (12). This is a standard convex optimization problem that can be solved numerically using a standard optimization technique such as SQP (Edgar & Himmelblau, 1989).

### 3.2. Internal stability under I/O linearizing state feedback

The diffeomorphism (5) decomposes the dynamics of the nonlinear system (1) into an external input/output part represented by a set of chain of integrators of the form of Eq. (8) and an internal unobservable part represented by Eq. (7). The application of the state feedback (13) to the nonlinear system (1) results in a set of $p$ linear subsystems. The I/O stability of the $v-y$ system is affected by these $p$ linear subsystems which can be designed for stability and performance using standard results from linear multivariable control theory (Skogestad & Postlethwaite, 1996). In addition, it is necessary to study the issue of internal stability, i.e., asymptotic stability of the states with respect to perturbations in initial conditions under no external input. This issue can be resolved by studying the problem of *zeroing the output*. The solution to this problem results in a dynamical system which describes the "internal" behavior of the system when the inputs and initial conditions are chosen in such a way so as to constrain the output vector to remain identically zero (Isidori, 1989). These dynamics are called *zero dynamics* of the system and their stability determine the internal stability of the closed-loop system under state feedback. Isidori (1989) developed a constructive procedure for representing the zero dynamics for square systems. We extend this procedure for nonsquare systems in Theorem 2.

**Theorem 2.** *The zero dynamics of nonsquare system are represented by*

$$\dot{\eta} = q(0,\eta) - p(0,\eta)[\beta(0,\eta)]^{\#}\alpha(0,\eta), \quad (18)$$

*where* $[\beta(0,\eta)]^{\#}$ *is the pseudo-inverse of* $\beta(0,\eta)$.

**Proof.** The proof is shown in the appendix. □

For internal stability of the I/O linearized system, the zero dynamics (18) should be stable. A nonlinear stability analysis of the zero dynamics is necessary to determine the region in state-space that correspond to asymptotically stable zero dynamics. Local results can be easily obtained by checking the eigenvalues of the linear approximation of the zero dynamics around the equilibrium.

## 4. Example

Consider a CSTR in which elementary isothermal liquid phase, series reactions are being carried out. The chemical reaction system is

$A + B \rightarrow C,$

$A + D \rightarrow E.$

The objective is to keep the concentrations of $C$ and $E$ at a desired set point by manipulating the inlet feed concentrations of $A$, $B$ and $D$. The system can be written in deviation form as

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2 + g_3(x)u_3, \quad (19)$$

$$y = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} \quad (20)$$

where

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} -k_1(x_1x_2 + x_1c_{BS} + x_2c_{AS}) - k_2(x_1x_4 + x_1c_{DS} + x_4c_{AS}) - \frac{F}{V}x_1 \\ -k_1(x_1x_2 + x_1c_{BS} + x_2c_{AS}) - \frac{F}{V}x_2 \\ k_1(x_1x_2 + x_1c_{BS} + x_2c_{AS}) - \frac{F}{V}x_3 \\ -k_2(x_1x_4 + x_1c_{DS} + x_4c_{AS}) - \frac{F}{V}x_4 \\ k_2(x_1x_4 + x_1c_{DS} + x_4c_{AS}) - \frac{F}{V}x_5 \end{bmatrix} \quad (21)$$

and

$$g = [g_1 \ g_2 \ g_3] = \begin{bmatrix} \frac{F}{V} & 0 & 0 \\ 0 & \frac{F}{V} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{F}{V} \\ 0 & 0 & 0 \end{bmatrix}. \quad (22)$$

The system parameters are $k_1 = 1$, $k_2 = 2$, $F = V = 3$. The desired operating steady state is $[0.89\ 0.53\ 0.47\ 0.36\ 0.64]^{\mathrm{T}}$. The steady-state input values are $[2\ 1\ 1]^{\mathrm{T}}$.

The procedure developed in the previous section is utilized to design the I/O linearizing controller. It is shown that it leads to economical control when compared to the a standard "squaring down" approach.

(1) The system has relative degree 2 with respect to both the outputs, $x_3$ and $x_5$.
(2) The system (19) under the transformation,

$$T(x) = \begin{bmatrix} x_1 \\ h_1(x) \\ L_f h_1(x) \\ h_2(x) \\ L_f h_2(x) \end{bmatrix} \tag{23}$$

gives

$$\dot{\xi}_1 = f_1|_{x=T^{-1}(\xi)} + \frac{F}{V}u_1 \quad \text{(zero dynamics)}, \tag{24}$$

$$\dot{\xi}_2 = \xi_3,$$

$$\dot{\xi}_3 = [L_f^2 x_3 + L_g L_f x_3 u]|_{x=T^{-1}(\xi)}, \tag{25}$$

$$\dot{\xi}_4 = \xi_5$$

$$\dot{\xi}_5 = [L_f^2 x_5 + L_g L_f x_5 u]|_{x=T^{-1}(\xi)}, \tag{26}$$

(3) The feedback law given by Eq. (13) with

$$\alpha = \begin{bmatrix} f_1 k_1(x_2 + C_{BS}) + f_2 k_1(x_1 + C_{AS}) - \frac{F}{V} f_3 \\ f_1 k_2(x_4 + C_{DS}) + f_4 k_2(x_1 + C_{AS}) - \frac{F}{V} f_5 \end{bmatrix} \tag{27}$$

and

$$\beta = \begin{bmatrix} \frac{F}{V}k_1(x_2 + C_{BS}) & \frac{F}{V}k_1(x_1 + C_{AS}) & 0 \\ \frac{F}{V}k_2(x_4 + C_{DS}) & 0 & \frac{F}{V}k_2(x_1 + C_{AS}) \end{bmatrix} \tag{28}$$

is used to linearize the system.

(4) In the outer loop, $v_1$ and $v_2$ are designed such that the poles of the linear subsystem are placed at $-4.5$.
(5) The optimization problem (17) is solved numerically at each time step in the inner loop.

Suppose that the inputs are weighted as follows:

$$W = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{29}$$

This means that the cost of the first input, species $A$, is two-and-half times as costly as species $D$ and the cost of species $B$ is five times as costly. The control law can be found by solving the optimization problem (17) using SQP with linear constraints. This controller is compared with the case where only two inputs are considered (squaring down procedure). The input and output profiles are shown in Figs. 1–4. It is observed that the outputs are driven to their steady-state values in all cases. It can be seen that in the case where all three inputs are utilized, $u_1$ and $u_2$ (the more expensive inputs) are used in lesser amounts than $u_3$. The objective values for the four different cases are given in Table 1.

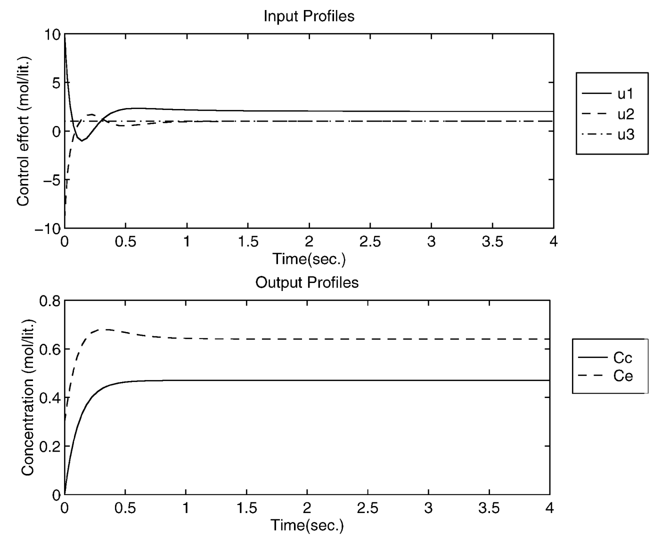


Fig. 1. Input and output profiles (using $u_1$ and $u_2$).

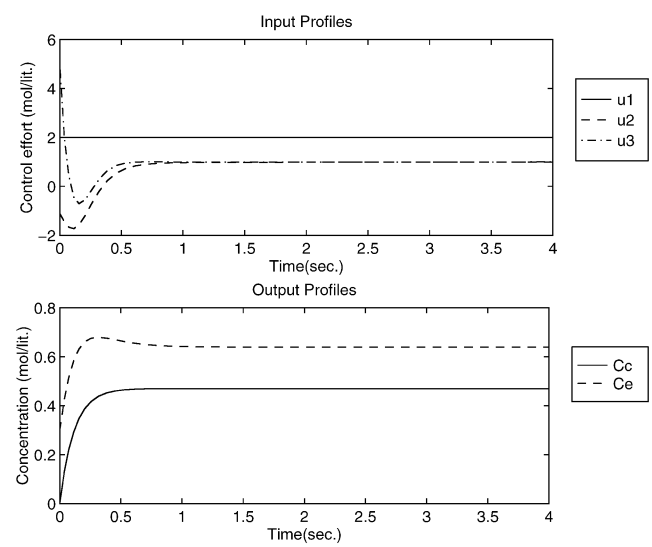


Fig. 2. Input and output profiles (using $u_2$ and $u_3$).

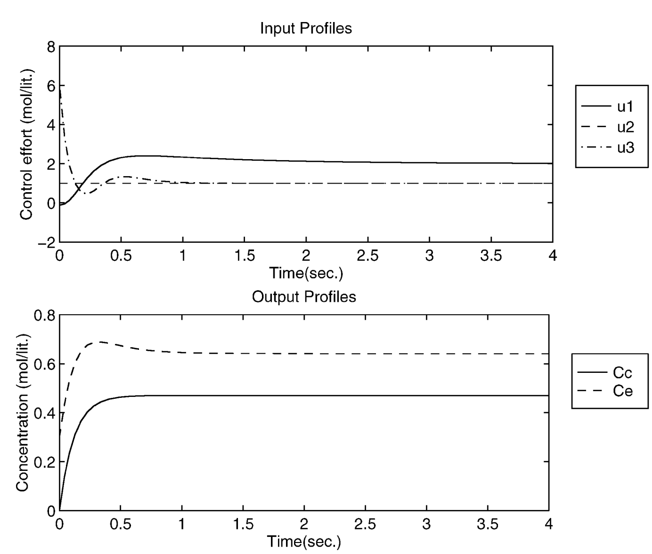


Fig. 3. Input and output profiles (using $u_1$ and $u_3$).

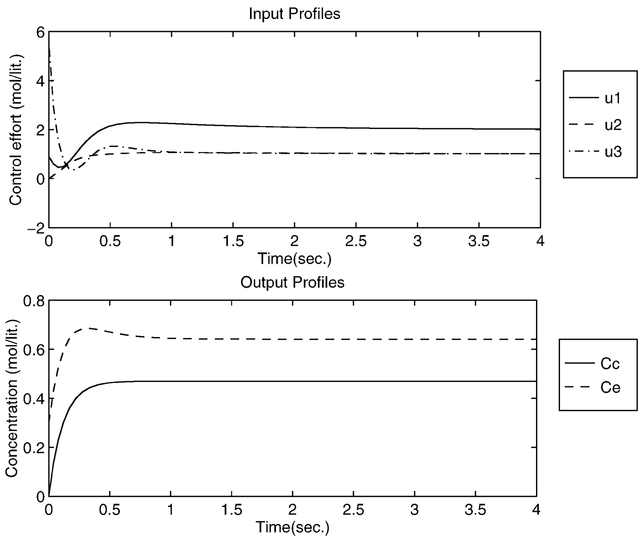


Fig. 4. Input and output profiles (using $u_1$, $u_2$ and $u_3$).

Table 1
Total cost for different input combinations

| Control inputs | Input cost |
|---|---|
| $u_1$ and $u_2$ | 17,125 |
| $u_2$ and $u_3$ | 4405 |
| $u_1$ and $u_3$ | 4221 |
| $u_1$, $u_2$ and $u_3$ | 3447 |

It is observed that the cost of utilizing all three inputs is less than the cost of using only two inputs ("squaring down" procedure).

## 5. Conclusions

In this paper, a nonlinear feedback law is synthesized which utilizes all the available inputs to linearize the system in an I/O sense and minimize the cost of the control effort. These objectives are achieved by solving a convex optimization problem on-line. The utilization of all the inputs provides the design flexibility to find the optimal choice of inputs. When the cost of all the inputs is the same, this procedure is equivalent to calculating the Moore–Penrose pseudo-inverse. When the number of inputs and outputs are equal, this procedure reduces to the results available in the literature for I/O linearization of square multivariable systems.

## Notation

| | |
|---|---|
| $x_1 \dots x_n$ | system states |
| $y_1 \dots y_p$ | outputs |
| $r_i$ | relative degree with respect to output $y_i$ |
| $u_1 \dots u_m$ | original inputs |
| $v_1 \dots v_p$ | external reference inputs |
| $u_{\min}, u_{\max}$ | bounds on the inputs |
| $w_1 \dots w_m$ | Costs of the inputs |
| $W$ | matrix of costs |
| $A^{-1}$ | inverse of the nonsingular matrix $A$ |
| $A^{\ddagger}$ | Moore–Penrose pseudo-inverse of $A$ |
| $A^{\#}$ | Moore–Penrose-like pseudo-inverse of $A$ |
| $k_1, k_2$ | reaction rate constants |
| $C_A \dots C_E$ | concentrations of the species |
| $C_{AS} \dots C_{ES}$ | steady-state concentrations |
| $F$ | flow rate into the reactor |
| $V$ | volume of the reactor |
| $L_f h$ | Lie derivative of $h$ along $f$ |
| $\|\cdot\|_2$ | 2-Norm of a vector |

*Greek letters*

| | |
|---|---|
| $\alpha_i$ | repeated lie derivative defined in Eq. (9) |
| $\beta$ | characteristic matrix defined by Eq. (4) |
| $\xi_i$ | states of the linear subsystem |
| $\eta_i$ | states of the zero dynamics |

## Appendix A. Proof of Theorem 2

The zero dynamics for nonsquare systems can be obtained by solving the problem of *Zeroing the output*, i.e., to find initial conditions and inputs consistent with the constraint that the outputs $y_i$ are identically zero for all time. This approach is along the lines of Isidori (1989), where the zero dynamics are obtained for square systems.

If $y_i(t) = 0$ for all $t$ then,

$$h_i(x) = L_f h_1(x) = \cdots = L_f^{r_i - 1} h_1(x) = 0, \quad i = 1, \dots, p,$$

i.e., $\xi(t) = 0$ for all $t$.

The $r_i$th derivatives of $y_i$ should also be zero,

$$\frac{d^{r_i} y_i}{dt^{r_i}} = \alpha(0,\eta) + \beta(0,\eta) u = 0. \tag{30}$$

The above equation can be solved for $u$, by solving the optimization problem (17). Thus, the solution to the above equation is

$$u = -[\beta(0,\eta)]^{\#} \alpha(0,\eta). \tag{31}$$

If the output $y(t)$ has to be 0 for all $t$, then the initial state of the system must be such that $\xi(0) = 0$ and $\eta(0) = \eta_o$, where $\eta_o$ can be arbitrarily chosen. According to the values of $\eta_o$, the input is represented by Eq. (31), where $\eta$ is the solution to the system of differential equations of the form

$$\dot{\eta} = q(0,\eta) - p(0,\eta)[\beta(0,\eta)]^{\#} \alpha(0,\eta) \tag{32}$$

with initial conditions $\eta(0) = \eta_o$. The dynamics (32) characterize the zero dynamics of the system. □

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