Symbol Error Probability of Low-Order Orthogonal Signalings in Rayleigh Fading With General Diversity Combining

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***Abstract*—New analytical expressions for the symbol error probability of 3-ary and 4-ary orthogonal signaling and 6-ary and 8-ary biorthogonal signaling with general diversity combining in Rayleigh fading are derived.**

***Index Terms*—Diversity, fading channels, maximal ratio combining, selection combining, symbol error probability.**

I. INTRODUCTION

THE symbol error probability (SEP) for coherent detection of M-ary orthogonal and M-ary biorthogonal signaling constellations in an additive white Gaussian noise (AWGN) channel are well known. However, to the best of the authors' knowledge analytical expressions for the SEPs of orthogonal signaling in Rayleigh fading with diversity do not exist for $M > 2$. In this letter, we derive analytical expressions for the SEPs of general diversity combining (GDC) [1] with 3-ary and 4-ary orthogonal and 6-ary and 8-ary biorthogonal signaling in Rayleigh fading. Note that GDC includes as special cases, maximal ratio combining (MRC), hybrid-selection/maximal ratio combining (H-S/MRC) and selection combining (SC).

This letter is organized as follows. In Section II, we introduce the system model and recall some previous results used in the sequel. New analytical expressions for the SEPs of low-order orthogonal and biorthogonal signaling in Rayleigh fading with GDC are derived in Section III. Some numerical results are given in Section IV.

II. SEP OF GDC SYSTEM

We consider a diversity system with N branches. Let γ_i denote the instantaneous signal-to-noise ratio (SNR) of the ith diversity branch defined as

$$\gamma_i = \alpha_i^2 \frac{E_s}{N_i} \tag{1}$$

where E_s is the average symbol energy, N_i is the one-sided noise power spectral density of the ith branch and α_i is the instantaneous fading envelope. We assume that the channel is a Rayleigh fading channel and hence, the probability density function (pdf) of the instantaneous branch SNR is given by [1]

$$f_{\gamma_i}(x) = \begin{cases} \dfrac{1}{\Gamma_i} e^{-x/\Gamma_i}, & 0 \le x < \infty \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $\Gamma_i = \mathbb{E}[\gamma_i]$ and $\mathbb{E}[\cdot]$ denotes the expectation operation. We further assume that the diversity branches are independent and have equal average SNR, i.e., $\Gamma_i = \Gamma$ and so $f_{\gamma_i}(x) = f(x)$. The instantaneous SNR of the GDC system is given by [1]

$$\gamma_{\text{GDC}} = \sum_{i=1}^{N} q_i \gamma_{(i)} \tag{3}$$

where $q_i \in \{0, 1\}$ and $\{\gamma_{(1)}, \ldots, \gamma_{(N)}\}$ are the ordered diversity branches, i.e., $\gamma_{(1)} > \cdots > \gamma_{(N)}$. In a slowly fading channel, the SEP for the GDC system can be obtained by averaging the SEP of the unfaded signal over the fading. Thus, the SEP of GDC can be written as

$$P_{e,\text{GDC}} = \mathbb{E}_{\gamma_{\text{GDC}}}\{\Pr(e|\gamma_{\text{GDC}})\} = \int_0^\infty \Pr(e|\gamma_{\text{GDC}}) f_{\gamma_{\text{GDC}}}(\gamma) d\gamma. \tag{4}$$

where $\Pr(e|\gamma_{\text{GDC}})$ is the conditional SEP given γ_{GDC} and $f_{\gamma_{\text{GDC}}}(\gamma)$ is the pdf of γ_{GDC}. It was shown in [1] that the SEP of GDC in Rayleigh fading with N independent and identically distributed branches is given by

$$P_{e,\text{GDC}} = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \Pr\left(e \Big| \sum_{n=1}^{N} b_n v_n\right) \prod_{n=1}^{N} f_{V_N}(v_n)\, dv_n \tag{5a}$$

where $\Gamma V_n/n$ is the normalized nth "virtual branch" that can be interpreted as the difference between the adjacent ordered instantaneous SNRs, and where $f_{V_N}(v_n)$ are given by

$$f_{V_n}(v) = \begin{cases} \exp(-v), & 0 < v < \infty \\ 0, & \text{otherwise} \end{cases} \tag{5b}$$

$$b_n = \frac{\Gamma}{n} \sum_{i=1}^{n} q_i. \tag{5c}$$

For MRC, (5c) simplifies to

$$b_n^{\text{MRC}} = \Gamma, \quad i = 1, \ldots, N. \tag{6}$$

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For H-S/MRC, where L branches with the largest SNR's from the available N branches are selected, the coefficients b_n in (5c) are defined as

$$b_n^{\mathrm{H-S/MRC}} = \begin{cases} \Gamma, & n = 1, \ldots, L \\ \dfrac{\Gamma L}{n}, & n = L+1, \ldots, N \end{cases} \tag{7}$$

and for SC, (5c) reduces to

$$b_n^{\mathrm{SC}} = \frac{\Gamma}{n}, \quad n = 1, \ldots, N. \tag{8}$$

III. SEP of Classes of Orthogonal Signaling With GDC

The SEP of 3-ary orthogonal signaling in AWGN channels is given as [2, eq.(13)]

$$P_{e,\mathrm{AWGN}}^{3-\mathrm{orth}}(\gamma) = \frac{1}{\pi}\int_0^{2\pi/3} e^{-\gamma/(2\sin^2(\theta))} d\theta. \tag{9}$$

Substituting (9) in (5a) and using (5b) and (5c), the SEP of GDC with 3-ary orthogonal signaling is derived as

$$\begin{aligned} P_{e,\mathrm{GDC}}^{3-\mathrm{orth}} =& \frac{1}{\pi}\int_0^\infty\int_0^\infty \cdots \int_0^\infty \int_0^{2\pi/3} e^{-\frac{b_1 v_1 + \cdots + b_N v_N}{2\sin^2(\theta)}} \\ & \times e^{-(v_1+\cdots+v_N)} d\theta dv_1 \cdots dv_N \\ =& \int_0^{2\pi/3} \left\{ \prod_{n=1}^N \int_0^\infty e^{-(\frac{b_n + 2\sin^2(\theta)}{2\sin^2(\theta)})v_n} dv_n \right\} d\theta \\ =& \frac{1}{\pi}\int_0^{2\pi/3} \prod_{n=1}^N \left(\frac{2\sin^2(\theta)}{2\sin^2(\theta) + b_n} \right) d\theta. \end{aligned} \tag{10}$$

To the best of the authors' knowledge (10) is a new result. For MRC, using (10) with [3, eq. (5A.17)], the SEP of 3-ary orthogonal signaling can be obtained in closed-form as

$$P_{e,\mathrm{MRC}}^{3-\mathrm{orth}} = g(\Gamma, 3) \tag{11a}$$

where $g(\Gamma, M)$ is defined as

$$\begin{aligned} g(\Gamma, M) =& \frac{M-1}{M} - \sqrt{\frac{\Gamma}{\Gamma+2}} \left(\frac{1}{2} + \frac{\tan^{-1}\alpha}{\pi} \right) \\ & \cdot \sum_{k=0}^{N-1} \binom{2k}{k} \frac{1}{(4+2\Gamma)^k} - \sqrt{\frac{\Gamma}{\Gamma+2}} \frac{\sin\left(\tan^{-1}\alpha\right)}{\pi} \\ & \cdot \sum_{k=1}^{N-1} \sum_{i=1}^{k} \frac{T_{ik}}{\left(1+\frac{\Gamma}{2}\right)^k} \left[\cos^{2(k-i)+1}\left(\tan^{-1}\alpha\right) \right] \\ \alpha =& \sqrt{\frac{\Gamma}{\Gamma+2}} \cot\left(\frac{\pi}{M}\right) \\ T_{ik} =& \frac{\binom{2k}{k}}{\binom{2(k-i)}{k-i} 4^i [2(k-i)+1]} \end{aligned} \tag{11b}$$

Similarly, substituting the SEP of 4-ary orthogonal signaling in AWGN channel given in [2, eq. (18)], using the alternative representation of the Gaussian tail integral [4, eq. (9)]

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} e^{-x^2/(2\sin^2(\delta))} d\delta \tag{12}$$

and after some mathematical manipulations, the SEP of GDC with 4-ary orthogonal signaling is derived as (13), shown at the bottom of this page. The result in (13) is new. In the case of MRC, using [3, eq. (5A.4a)], (13) specializes to

$$\begin{aligned} P_{e,\mathrm{MRC}}^{4-\mathrm{orth}} =& J_N\left(\frac{3\Gamma}{4}\right) + \int_{\pi/6}^{5\pi/6} \frac{3\sin(\theta)}{2\pi\sqrt{2+\sin(\theta)}} \\ & \times A(\theta, \Gamma) \left[1 - J_N\left(\frac{3\sin^2(\theta)\Gamma}{8 + 4\sin^2(\theta) + 6\Gamma} \right) \right] \end{aligned} \tag{14a}$$

where

$$A(\theta, \Gamma) = \left(\frac{4 + 2\sin^2(\theta)}{4 + 2\sin^2(\theta) + 3\Gamma} \right)^N \tag{14b}$$

and $J_m(c)$ is a closed-form expression defined in [3, eq. (5A.4a)]. Note that the double integral in (13) is reduced to a single (14) in the case of MRC. The SEPs of the GDC system with 6-ary and 8-ary biorthogonal signaling can be derived in a manner similar to that used for 4-ary orthogonal signaling; the results are shown in (15) and (16) at the bottom of the next page. respectively. To the best of the author's knowledge, (15) and (16) are new results. For 6-ary orthogonal signaling with MRC, (15) can be simplified to

$$\begin{aligned} P_{e,\mathrm{MRC}}^{6-\mathrm{biorth}} =& J_N(\Gamma) + \frac{4}{\pi}\int_0^{\pi/4} \frac{\cos(\theta)}{\sqrt{1+\cos^2(\theta)}} \\ & \times \left[\left(\frac{1+\cos^2\theta}{\sqrt{1+\cos^2\theta}} \right)^N - J_N\left(\frac{3\sin^2(\theta)\Gamma}{8+4\sin^2(\theta)} \right) \right] d\theta. \end{aligned} \tag{17}$$

Using [3, eq. (5A.17)], the SEP of 8-ary biorthogonal signaling with MRC can be obtained from (16) as (18) shown at the bottom of the next page, where $g(\Gamma, M)$ is defined in (11b). Note that for the no diversity case, i.e., $N = 1$, the results in (11), (14), (17) and (18) reduce to [2, eqs. (14), (19), (26), (28)], respectively, as expected.

IV. Numerical Examples

The SEP of SC, H-S/MRC and MRC for 3-ary and 4-ary orthogonal signaling and 6-ary and 8-ary biorthogonal signaling can be derived from (10), (13), (15) and (16), by substituting the corresponding coefficients b_n^{SC}, $b_n^{\mathrm{H-S/MRC}}$ and b_n^{MRC}, respectively. The SEP of 4-ary orthogonal signaling is plotted in Fig. 1

$$\begin{aligned} P_{e,\mathrm{GDC}}^{4-\mathrm{orth}} =& \frac{1}{\pi}\int_0^{\pi/2} \prod_{n=1}^N \left(\frac{4\sin^2(\delta)}{4\sin^2(\delta) + 3b_n} \right) d\delta + \int_{\pi/6}^{5\pi/6} \frac{3\sin(\theta)}{2\pi\sqrt{2+\sin^2(\theta)}} \prod_{n=1}^N \left(\frac{4+2\sin^2(\theta)}{4+2\sin^2(\theta)+3b_n} \right) d\theta \\ & - \frac{1}{\pi}\int_{\pi/6}^{5\pi/6}\int_0^{\pi/2} \frac{3\sin(\theta)}{2\pi\sqrt{2+\sin^2(\theta)}} \times \prod_{n=1}^N \left(\frac{2\sin^2(\delta)\left(4+2\sin^2(\theta)\right)}{2\sin^2(\delta)\left(4+2\sin^2(\theta)\right) + 3\left(\sin^2(\theta) + 2\sin^2(\delta)\right) b_n} \right) \times d\delta d\theta. \end{aligned} \tag{13}$$

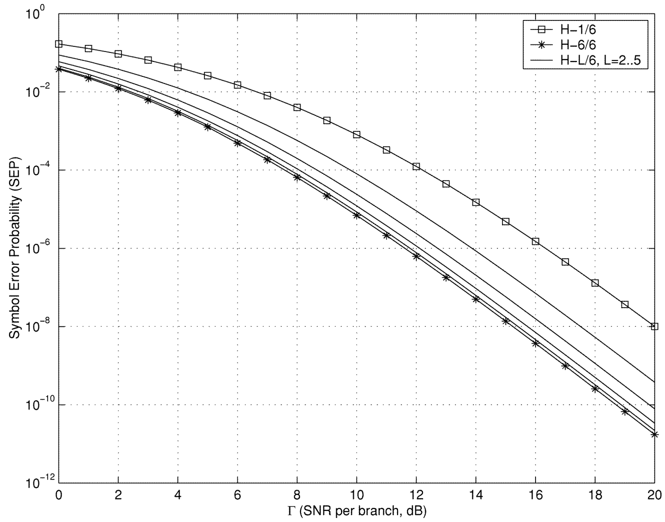

Fig. 1. The SEP for coherent detection of 4-ary orthogonal signaling with H-S/MRC as a function of average SNR per branch for $L = 1, \ldots, 6$ and $N = 6$. The curves are distinguished by different H-L/6 starting from the highest curve representing H-1/6 and decrease monotonically to the lowest curve representing H-6/6.

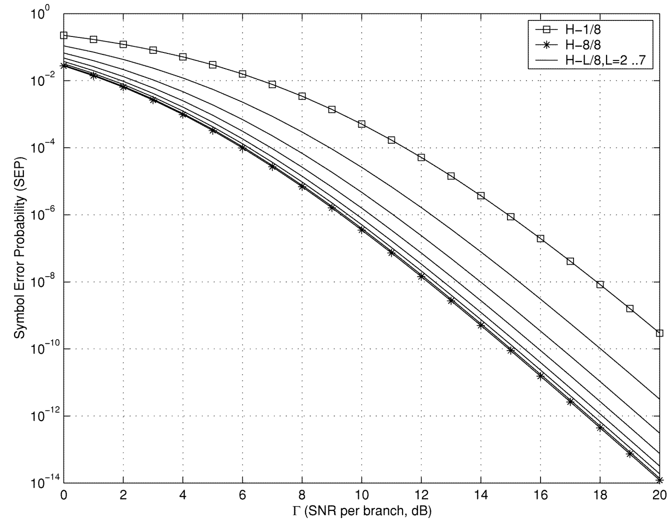

Fig. 2. The SEP for coherent detection of 8-ary biorthogonal signaling with H-S/MRC as a function of average SNR per branch for $L = 1, \ldots, 8$ and $N = 8$. The curves are distinguished by different H-L/8 starting from the highest curve representing H-1/8 and decrease monotonically to the lowest curve representing H-8/8.

for $L = 1$,…, 6 and $N = 6$. Fig. 1 shows that most of the gain of MRC is achieved by H-S/MRC for $L = 4$ and 5, losing 0.447 and 0.131 dB when compared to full MRC with $L = 6$ at a SEP of 10^{-4}. The losses are greater for $L = 2$ and 3, being 2.160 and 1.039 dB, respectively. The SEP of 8-ary biorthogonal signaling is plotted in Fig. 2 for $L = 1, \ldots, 8$ and $N = 8$ and similar to Fig. 1 most of the gain of MRC is achieved by H-S/MRC for $L = 5, 6,$ and 7. The losses are 0.506, 0.227and 0.070 dB, respectively. For small values of L, the losses at this SEP are 2.942, 1.694, and 0.970 dB for $L = 2, , 3,$ and 4, respectively. Figs. 1 and 2 show that for these low-order orthogonal signalings H-S/MRC can achieve performance close to that of MRC, with reduced complexity.

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$$P_{e,\mathrm{GDC}}^{6-\mathrm{biorth}} = \frac{1}{\pi}\int_0^{\pi/2}\prod_{n=1}^{N}\left(\frac{\sin^2(\delta)}{\sin^2(\delta)+b_n}\right)d\delta + \int_0^{\pi/4}\frac{4\cos(\theta)}{\pi\sqrt{1+\cos^2(\theta)}}\times\prod_{n=1}^{N}\left(\frac{1+\cos^2(\theta)}{1+\cos^2(\theta)+b_n}\right)d\theta - \frac{4}{\pi^2}\int_0^{\pi/4}\int_0^{\pi/2}\frac{\cos(\theta)}{\sqrt{1+\cos^2(\theta)}}\times\prod_{n=1}^{N}\left(\frac{\sin^2(\delta)\left(1+\cos^2(\theta)\right)}{\sin^2(\delta)\left(1+\cos^2(\theta)\right)+\cos^2(\theta)b_n}\right)d\delta d\theta \quad (15)$$

$$P_{e,\mathrm{GDC}}^{8-\mathrm{biorth}} = \frac{3}{\pi}\int_0^{3\pi/4}\prod_{n=1}^{N}\left(\frac{2\sin^2(\theta)}{2\sin^2(\theta)+b_n}\right)d\theta - \frac{12}{\pi^2}\int_{\pi/4}^{\pi}\int_0^{\pi/4}\frac{\cos^2(\theta_2)}{\cos^2(\theta_2)+\sin^2(\theta_1)} \times\prod_{n=1}^{N}\left(\frac{2\cos^2(\theta_2)\sin^2\left(\theta_1-\frac{\pi}{4}\right)}{2\cos^2(\theta_2)\sin^2\left(\theta_1-\frac{\pi}{4}\right)+\left(\cos^2(\theta_2)+\sin^2(\theta_1)\right)b_n}\right)\times d\theta_2 d\theta_1 \quad (16)$$

$$P_{e,\mathrm{MRC}}^{8-\mathrm{biorth}} = 3g(\Gamma,4) - \frac{12}{\pi^2}\int_{\pi/4}^{\pi}\int_0^{\pi/4}\frac{\cos^2(\theta_2)}{\cos^2(\theta_2)+\sin^2(\theta_1)}\left(\frac{2\cos^2(\theta_2)\sin^2\left(\theta_1-\frac{\pi}{4}\right)}{2\cos^2(\theta_2)\sin^2\left(\theta_1-\frac{\pi}{4}\right)+\left(\cos^2(\theta_2)+\sin^2(\theta_1)\right)\Gamma}\right)^N d\theta_1 d\theta_2 \quad (18)$$