Large Penalty of Hybrid Diversity with Uncoded Modulation in Slow Rayleigh Fading

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***Abstract*— In hybrid selection/maximal ratio combining (H-S/MRC)the receiver selects the L branches with the largest signal-to-noise ratios (SNRs) from N available diversity branches and performs maximal ratio combining (MRC). A penalty is incurred with respect to MRC which can be defined as the increase in SNR required for H-S/MRC to achieve the same symbol error probability as MRC. Recently, this penalty was investigated specifically for two-dimensional modulations. In this paper, we derive the asymptotic SNR penalties in slow Rayleigh fading for small and large values of SNR for general uncoded signaling constellations.**

***Index Terms*— Diversity combining, error probability, fading channel, hybrid selection/maximal ratio combining, maximal ratio combining, selection combining.**

I. Introduction

In hybrid selection/maximal ratio diversity combining (H-S/MRC) the receiver uses the L branches with the largest signal-to-noise ratios (SNR) from the available N branches and performs maximal ratio combining (MRC) on the selected branches. Since all the available N branches are not used in H-S/MRC, a penalty is incurred compared to MRC. The penalty is defined as the increase in SNR required for H-S/MRC to achieve the same performance as that of MRC [1]. In mathematical terms, the SNR penalty, β, is defined such that

$$P_{e,\text{H-S/MRC}}(\beta\Gamma) = P_{e,\text{MRC}}(\Gamma) \tag{1}$$

where $P_{e,\text{MRC}}(\cdot)$ and $P_{e,\text{H-S/MRC}}(\cdot)$ are the symbol error probabilities (SEPs) of MRC and H-S/MRC, respectively and Γ is the average SNR for each branch. Solving for β in (1) we obtain

$$\beta(\Gamma) = \frac{1}{\Gamma} P^{-1}_{e,\text{H-S/MRC}}\{P_{e,\text{MRC}}(\Gamma)\} \tag{2}$$

which is, in general, a function of Γ. In [1], small SNR and large SNR asymptotes are derived for the SNR penalty of M-ary phase shift keying (MPSK) in Rayleigh fading. Also, simple lower and upper bounds for the penalty of H-S/MRC were derived in [1]. The results of [1] were generalized to any two-dimensional signaling constellation with polygonal decision regions in [2],[3]. In this paper, we show that the penalties approach asymptotes for small SNR and for large SNR for any modulation format in slow flat Rayleigh fading channels. Analytical expressions are derived for small and large asymptotes. The method used here to derive these asymptotes is different from the method used in [1]-[3] and is not a generalization of the method used in [1]-[3]. This method lends itself to general modulation formats whereas the method in [1]-[3] can only be used for one- and two-dimensional modulation formats. We show that the SNR penalty is, in general, a function of the modulation order, modulation format and the average SNR, but the *asymptotic* SNR penalties are independent of the modulation order and the modulation format and are only a function of L and N.

This paper is organized as follows. In Section II, the system model is presented. The asymptotic values for the penalty of hybrid diversity for small and large values of SNR are derived in Section III and IV, respectively. Some numerical examples are given in Section V; the conclusion is given in Section VI.

II. System Model

The system model we consider is a slow flat Rayleigh fading channel. We consider N independent branches each with instantaneous SNR of the form

$$\gamma_i = \alpha_i^2 \frac{E_s}{N_{0i}}, \quad i = 1, \cdots, N \tag{3}$$

where E_s is the average symbol energy, N_{0i} is the two-sided noise power spectral density of the i^{th} branch and α_i^2 has a probability density function (pdf) of the form

$$f_{\gamma_i}(x) = \frac{1}{\Gamma_i} \exp\left(-\frac{x}{\Gamma_i}\right). \tag{4}$$

We assume further that the γ_i's have equal average SNR, i.e., $\Gamma_i = \Gamma$, $i = 1, \cdots, N$. We consider a general diversity combining system (GDC) given in [4] with instantaneous SNR of the form

$$\gamma_{\text{GDC}} = \sum_{i=1}^{N} g_i \gamma_{(i)} \tag{5}$$

where $g_i \in \{0, 1\}$ and and $\{\gamma_{(1)}, \cdots, \gamma_{(N)}\}$ are the ordered diversity branches, i.e., $\gamma_{(1)} > \cdots > \gamma_{(N)}$. A convenient expression for the SEP of GDC in Rayleigh fading with N independent and identically distributed channels is given in [4] as

$$P_{e,\text{GDC}} = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \Pr\left(e|\Gamma \sum_{n=1}^{N} b_n v_n\right) \times \prod_{n=1}^{N} f_{V_n}(v_n) dv_1 \cdots dv_n \tag{6a}$$

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where $\Pr(e|\Gamma \sum_{n=1}^{N} b_n v_n)$ is the conditional error probability in an additive white Gaussian noise (AWGN) channel,

$$f_{V_n}(v) = \begin{cases} \exp(-v), & 0 < v < \infty \\ 0, & \text{otherwise} \end{cases} \tag{6b}$$

and

$$b_n = \frac{1}{n}\sum_{i=1}^{n} g_i. \tag{6c}$$

For MRC, (6c) simplifies to

$$b_n^{\text{MRC}} = 1, \quad n = 1, \cdots, N \tag{7}$$

and in the case of H-S/MRC, b_n is given by

$$b_n^{\text{H-S/MRC}} = \begin{cases} 1 & , \quad n = 1, \cdots, L \\ \frac{L}{n} & , \quad n = L+1, \cdots, N. \end{cases} \tag{8}$$

III. Asymptotic Penalty for Small SNR

We assume that $\Pr(e|\Gamma \sum_{n=1}^{N} b_n v_n)$ in (6a) has a series expansion in terms of $\sqrt{\Gamma}$. This is a reasonable assumption because the SEP of any uncoded modulation in an AWGN channel is a function of the distances between the signal points, which are proportional to the square root of the SNR. Thus, $\Pr(e|\Gamma \sum_{n=1}^{N} b_n v_n)$ can be written as [1]

$$\Pr\left(e|\Gamma\sum_{n=1}^{N} b_n v_n\right) = \sum_{i=0}^{\infty} c_i\left(\sqrt{\Gamma(b_1 v_1 + \cdots + b_N v_N)}\right)^i \tag{9}$$

[1]This equality assumes the existence of a series expansion in terms of $\sqrt{\Gamma}$. We assume this existence as plausible and do not prove it.

where c_i are the infinite series coefficients. Substituting (9) in (6a) and using (6b) and (6c), one obtains formally

$$P_{e,\text{GDC}} = \sum_{i=0}^{\infty} c_i \sqrt{\Gamma^i}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\sqrt{(b_1 v_1 + \cdots + b_N v_N)^i} \times e^{-(v_1+\cdots+v_N)} dv_1 \cdots dv_N. \tag{10}$$

For small values of SNR, Γ, the infinite series in (10) can be approximated by the first two terms as

$$P_{e,\text{GDC}} \cong c_0 + c_1\sqrt{\Gamma}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\sqrt{b_1 v_1 + \cdots + b_N v_N} \times e^{-(v_1+\cdots+v_N)} dv_1 \cdots dv_N. \tag{11}$$

To complete the analysis, we require two ancillary results. These are stated here as Lemma 1 and Lemma 2.

Lemma 1: Define

$$I_1(x_1, \cdots, x_N) \triangleq \int_0^{\infty}\cdots\int_0^{\infty}\sqrt{x_1 v_1 + \cdots + x_N v_N} \times e^{-(v_1+\cdots+v_N)} dv_1 \cdots dv_N \tag{12}$$

and $I_2(x_1, \cdots, x_N)$ given in (13) at the bottom of the page. Then

$$I_1(x_1, \cdots, x_N) = I_2(x_1, \cdots, x_N) \tag{14}$$

for any integer $N \geq 1$ and for all $x_i > 0, \quad i = 1, \cdots, N$.

Proof: The proof is by induction. For $N = 1$, both integrals reduce to $\frac{\sqrt{\pi x_1}}{2}$ using standard integration methods. Now assume that (14) holds for $N = k-1$, that is

$$I_1(x_1, \cdots, x_{k-1}) = I_2(x_1, \cdots, x_{k-1}). \tag{15}$$

We now show that (14) is also valid for $N = k$. We start by writing $I_1(x_1, \cdots, x_N)$ as shown in (16c) at the

$$\begin{aligned} I_2(x_1, \cdots, x_N) \triangleq & \frac{\sqrt{x_1}}{\sqrt{\pi}}\int_0^{\infty}\left(\frac{x_1}{x_1 + x_2 u^2}\right)\left(\frac{x_1}{x_1 + x_3 u^2}\right)\cdots\left(\frac{x_1}{x_1 + x_N u^2}\right)\left(\frac{1}{1+u^2}\right) du \\ & + \frac{\sqrt{x_2}}{\sqrt{\pi}}\int_0^{\infty}\left(\frac{x_2}{x_2 + x_3 u^2}\right)\left(\frac{x_2}{x_2 + x_4 u^2}\right)\cdots\left(\frac{x_2}{x_2 + x_N u^2}\right)\left(\frac{1}{1+u^2}\right) du + \cdots + \frac{\sqrt{x_N}}{\sqrt{\pi}}\int_0^{\infty}\frac{1}{1+u^2} du. \end{aligned} \tag{13}$$

$$I_1(x_1, \cdots, x_k) = \int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\sqrt{x_1 v_1 + \cdots + x_k v_k}\, e^{-(v_1+\cdots+v_k)} dv_1 \cdots dv_k \tag{16a}$$

$$\begin{aligned} = & \frac{\sqrt{x_1 \pi}}{2}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} e^{\frac{x_2 v_2 + \cdots + x_k v_k}{x_1} - (v_2+\cdots+v_k)} \operatorname{erfc}\left(\sqrt{\frac{x_2 v_2 + \cdots + x_k v_k}{x_1}}\right) dv_2 \cdots dv_k \\ + & \int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\sqrt{x_2 v_2 + \cdots + x_k v_k}\, e^{-(v_2+\cdots+v_k)} dv_2 \cdots dv_k \end{aligned} \tag{16b}$$

$$= G_1(x_2, \cdots, x_k) + I_1(x_2, \cdots, x_k) = G_1(x_2, \cdots, x_k) + I_2(x_2, \cdots, x_k) \tag{16c}$$

$$G_1(x_2, \cdots, x_k) = \frac{\sqrt{x_1 \pi}}{2}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} e^{\frac{x_2 v_2 + \cdots + x_k v_k}{x_1} - (v_2+\cdots+v_k)} \operatorname{erfc}\left(\sqrt{\frac{x_2 v_2 + \cdots + x_k v_k}{x_1}}\right) dv_2 \cdots dv_k \tag{16d}$$

$$= \frac{\sqrt{x_1 \pi}}{2}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} e^{\frac{x_2 v_2 + \cdots + x_k v_k}{x_1} - (v_2+\cdots+v_k)}\left\{\frac{2}{\pi}\int_0^{\pi/2} e^{-\frac{x_2 v_2 + \cdots + x_k v_k}{x_1 \sin^2(\theta)}} d\theta\right\} dv_2 \cdots dv_k. \tag{16e}$$

bottom of the previous page where $G_1(x_2, \cdots, x_k)$ is defined in (16d). In obtaining (16e) from (16d), Craig's alternative representation of the complementary error function [5] has been used. Changing the order of integration, using the change of variables $u = \cot(\theta)$ in (16e), and again using Craig's representation yields (17).

Note that the interchange of the order of integration in (17) is valid because the integrand is a continuous function in the region of integration [6, Fubini's Theorem, p. 394]. Now, substituting (17c) in (16c) gives $I_1(x_1, \cdots, x_k) = I_2(x_1, \cdots, x_k)$ and, hence, the proof is complete. ∎

Lemma 2: Define

$$I_3(x_1, \cdots, x_N) \triangleq \frac{1}{\sqrt{\pi}} \times \int_0^\infty \left(1 - \frac{u^{2N}}{(u^2 + x_1)\cdots(u^2 + x_N)}\right) du. \tag{18}$$

Then

$$I_3(x_1, \cdots, x_N) = I_2(x_1, \cdots, x_N) \tag{19}$$

for $N \geq 1$ and $x_i > 0, i = 1, \cdots, N$, where $I_2(x_1, \cdots, x_N)$ is defined in (13).

Proof: Mathematical induction is used to prove (19). For $N = 1$, it is easy to verify that $I_3(x_1) = I_2(x_1) = \frac{\sqrt{\pi x_1}}{2}$ using standard integration methods. Assuming that (19) is valid for $N = k - 1$, i.e.,

$$I_3(x_1, \cdots, x_{k-1}) = I_2(x_1, \cdots, x_{k-1}), \tag{20}$$

we show that (19) is also valid for $N = k$. That is

$$I_3(x_1, \cdots, x_k) = I_2(x_1, \cdots, x_k). \tag{21}$$

The left side and the right side of (21) can be written as (22c) and (23), respectively, shown at the bottom of the page. Using the change of variable $u = \frac{\sqrt{x_1}}{v}$ in (23) we obtain

$$I_2(x_1, \cdots, x_k) = I_2(x_2, \cdots, x_k) + \frac{1}{\sqrt{\pi}} \times \int_0^\infty \frac{x_1 v^{2(k-1)}}{(x_1 + v^2)(x_2 + v^2)\cdots(x_k + v^2)} dv. \tag{24}$$

Comparing (22c) with (24) and using (20) gives (21) and, hence, the proof is complete. ∎

Using the results of Lemma 1 and Lemma 2 we can show that

$$\int_0^\infty \cdots \int_0^\infty \sqrt{b_1 v_1 + \cdots + b_N v_N} e^{-(v_1 + \cdots + v_N)} dv_1 \cdots dv_N = \frac{1}{\sqrt{\pi}} \int_0^\infty \left(1 - \frac{u^{2N}}{(u^2 + b_1)\cdots(u^2 + b_N)}\right) du. \tag{25}$$

Substituting (25) in (11), we obtain a simplified expression for the SEP of a GDC system for small values of SNR as

$$P_{e,\text{GDC}} \cong c_0 + \frac{c_1\sqrt{\Gamma}}{\sqrt{\pi}} \int_0^\infty \left(1 - \frac{u^{2N}}{(u^2 + b_1)\cdots(u^2 + b_N)}\right) du. \tag{26}$$

Consequently, by substituting (7) and (8) in (26) we obtain the SEP for MRC and H-S/MRC for small values of SNR, as

$$P_{e,\text{MRC}} \cong c_0 + \frac{c_1\sqrt{\Gamma}}{\sqrt{\pi}} \int_0^\infty \left\{1 - \left[\frac{u^2}{u^2 + 1}\right]^N\right\} du \tag{27}$$

and

$$P_{e,\text{H-S/MRC}} \cong c_0 + \frac{c_1\sqrt{\Gamma}}{\sqrt{\pi}} \times \int_0^\infty \left\{1 - \left[\frac{u^2}{u^2 + 1}\right]^L \prod_{n=L+1}^{N} \left[\frac{u^2}{\frac{L}{n} + u^2}\right]\right\} du \tag{28}$$

$$\begin{aligned}
G_1 &= \sqrt{\frac{x_1}{\pi}} \int_0^{\pi/2} \int_0^\infty e^{-\frac{x_1\sin^2(\theta) + x_2\cos^2(\theta)}{x_1\sin^2(\theta)} v_2} dv_2 \cdots \int_0^\infty e^{-\frac{x_1\sin^2(\theta) + x_k\cos^2(\theta)}{x_1\sin^2(\theta)} v_k} dv_k d\theta && \text{(17a)} \\
&= \sqrt{\frac{x_1}{\pi}} \int_0^{\pi/2} \frac{x_1\sin^2(\theta)}{x_1\sin^2(\theta) + x_2\cos^2(\theta)} \cdots \frac{x_1\sin^2(\theta)}{x_1\sin^2(\theta) + x_k\cos^2(\theta)} d\theta && \text{(17b)} \\
&= \sqrt{\frac{x_1}{\pi}} \int_0^\infty \frac{x_1}{(x_1 + x_2u^2)} \cdots \frac{x_1}{(x_1 + x_ku^2)} \frac{1}{(1 + u^2)} du. && \text{(17c)}
\end{aligned}$$

$$\begin{aligned}
I_3(x_1, \cdots, x_k) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \left(1 - \frac{u^{2k}}{(u^2 + x_1)(u^2 + x_2)\cdots(u^2 + x_k)}\right) du && \text{(22a)} \\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \left(1 - \frac{u^{2(k-1)}}{(u^2 + x_2)(u^2 + x_3)\cdots(u^2 + x_k)}\right) du \\
&+ \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{x_1 u^{2(k-1)}}{(x_1 + u^2)(x_2 + u^2)\cdots(x_k + u^2)} du && \text{(22b)} \\
&= I_3(x_2, \cdots, x_k) + \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{x_1 u^{2(k-1)}}{(x_1 + u^2)(x_2 + u^2)\cdots(x_k + u^2)} du. && \text{(22c)}
\end{aligned}$$

$$I_2(x_1, \cdots, x_k) = I_2(x_2, \cdots, x_k) + \sqrt{\frac{x_1}{\pi}} \int_0^\infty \frac{x_1}{(x_1 + x_2u^2)} \frac{x_1}{(x_1 + x_3u^2)} \cdots \frac{x_1}{(x_1 + x_ku^2)} \frac{1}{(1 + u^2)} du. \tag{23}$$

respectively. Substituting (27) and (28) in (1) and solving for the SNR penalty β, one obtains the asymptotic SNR penalty for small values of SNR, $\beta_{\mathrm{L}}^{\mathrm{A}}$, as

$$\beta_{\mathrm{L}}^{\mathrm{A}} = \left[\frac{K(N,N)}{K(L,N)}\right]^2 \tag{29a}$$

where

$$K(L,N) \triangleq \int_0^\infty \left\{1 - \left[\frac{u^2}{u^2+1}\right]^L \prod_{n=L+1}^{N} \left[\frac{u^2}{\frac{L}{n}+u^2}\right]\right\} du. \tag{29b}$$

Note that the asymptotic SNR penalty derived in (29a) is equal to the asymptotic SNR penalty derived in [1] and [3] for MPSK and generalized two-dimensional signaling, respectively. Our derivation here proves that the same small SNR penalty is incurred for all modulation formats, depending only on L and N.

IV. Asymptotic Penalty for Large SNR

In this section, we give the asymptotic SNR penalty for large values of SNR. Using [7, Theorem 1], one can show that for large values of SNR, $P_{e,\mathrm{GDC}}$ can be written as

$$P_{e_{asy}}^{\mathrm{GDC}} = \frac{1}{\Gamma^N \prod_{i=1}^{N} b_i} \times \int_0^\infty \int_0^\infty \cdots \int_0^\infty \Pr\left(e|\sum_{i=1}^{N} u_i\right) du_1 \cdots du_N. \tag{30}$$

Consequently, the asymptotic SEP for MRC and H-S/MRC can be expressed as

$$P_{e_{asy}}^{\mathrm{MRC}} = \frac{1}{\Gamma^N} \int_0^\infty \int_0^\infty \cdots \int_0^\infty \Pr\left(e|\sum_{i=1}^{N} u_i\right) du_1 \cdots du_N \tag{31}$$

and

$$P_{e_{asy}}^{\text{H-S/MRC}} = \frac{N!}{L!L^{N-L}\Gamma^N} \times \int_0^\infty \int_0^\infty \cdots \int_0^\infty \Pr\left(e|\sum_{i=1}^{N} u_i\right) du_1 \cdots du_N \tag{32}$$

respectively. Substituting (31) and (32) in (1) and solving for β, the asymptotic SNR penalty for large SNR, $\beta_{\mathrm{U}}^{\mathrm{A}}$, is obtained as

$$\beta_{\mathrm{U}}^{\mathrm{A}} = \left(\frac{N!}{L!L^{N-L}}\right)^{\frac{1}{N}}. \tag{33}$$

Note that the asymptotic SNR penalty for large values of SNR, derived in (33) is the same as the asymptotic SNR penalty for MPSK for large values of SNR presented in [1] and is also equal to the SNR penalty upper bound derived in [2] and [3] for two-dimensional signaling with polygonal decision regions. The SNR penalty for large SNR presented in (33) is independent of the modulation format and depends on L and N only.

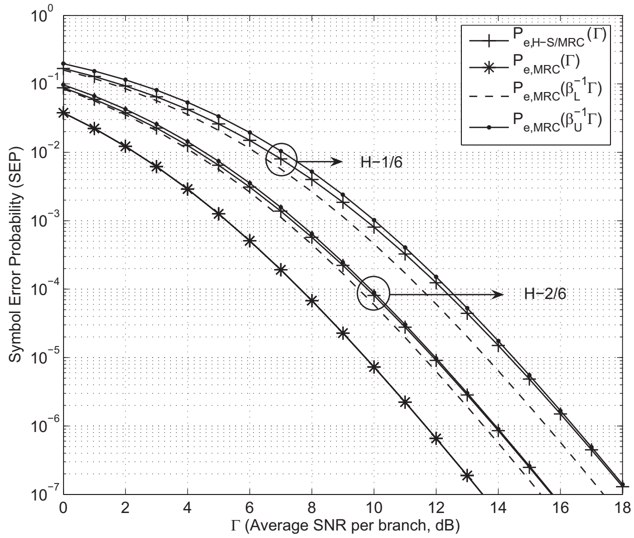

Fig. 1. The symbol error probability as a function of average SNR per branch for coherent detection of 4-ary orthogonal signaling with H-S/MRC for $L = 1, 2$ and 6. The conjectured lower and upper bounds are plotted for comparison.

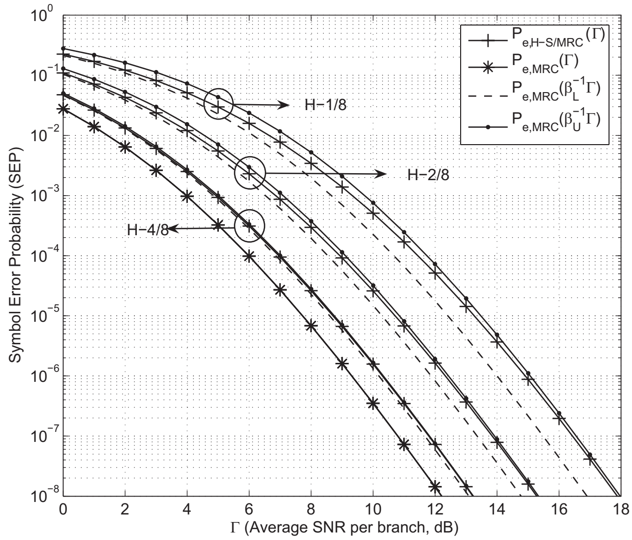

Fig. 2. The symbol error probability as a function of average SNR per branch for coherent detection of 8-ary biorthogonal signaling with H-S/MRC for $L = 1, 2, 4$ and 8. The conjectured lower and upper bounds are plotted for comparison.

V. Numerical Examples

In this section, some numerical examples are presented. Based on the examples, we also conjecture that previously published lower and upper bounds to the SNR penalty, valid for two-dimensional signaling with polygonal decision regions, are also valid for other modulation formats. In [2], a lower bound was derived to the SNR penalty of hybrid diversity with two-dimensional signaling as

$$\beta_{\mathrm{L}} = \frac{N}{L(1 + \sum_{n=L+1}^{N} \frac{1}{n})}. \tag{34}$$

In the examples, we will see that (34) appears also to be a bound to the SNR penalty of hybrid diversity for higher-

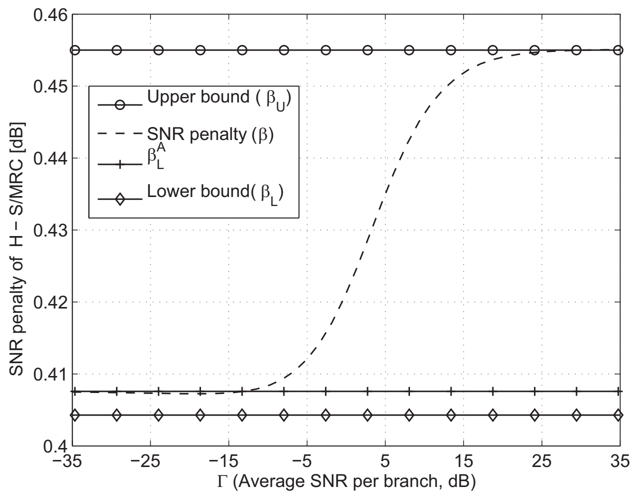

Fig. 3. The SNR penalty as a function of SNR for 4-ary orthogonal signaling with H-S/MRC for $(L, N) = (4, 6)$. The conjectured lower and upper bounds are plotted for comparison.

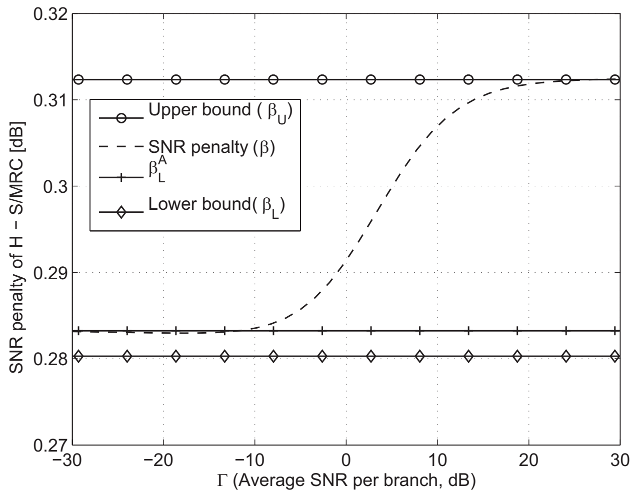

Fig. 4. The SNR penalty as a function of SNR for 4-ary orthogonal signaling with H-S/MRC for $(L, N) = (3, 4)$. The conjectured lower and upper bounds are plotted for comparison.

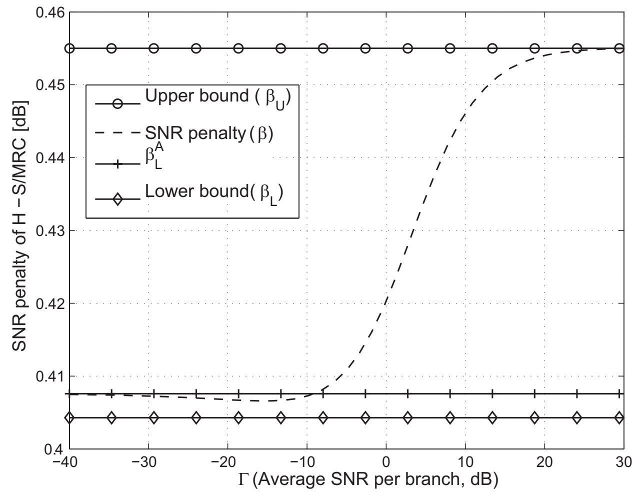

Fig. 5. The SNR penalty as a function of SNR for 8-ary biorthogonal signaling with H-S/MRC for $(L, N) = (4, 6)$. The conjectured lower and upper bounds are plotted for comparison.

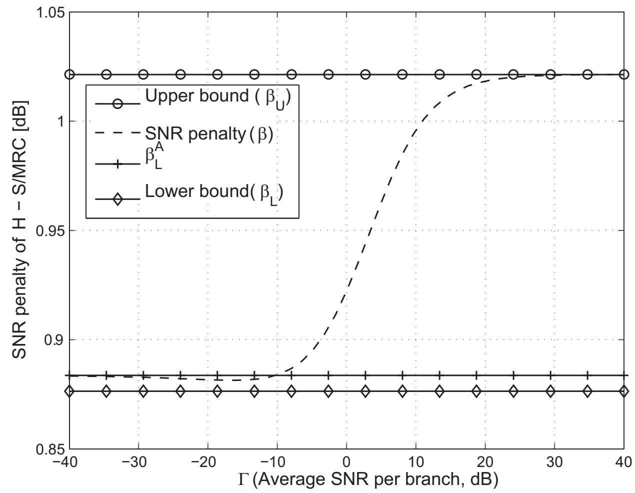

Fig. 6. The SNR penalty as a function of SNR for 8-ary biorthogonal signaling with H-S/MRC for $(L, N) = (4, 8)$. The conjectured lower and upper bounds are plotted for comparison.

order modulation formats. Also, an upper bound to the SNR penalty valid for two-dimensional signaling constellations with polygonal decision regions was derived in [2] as

$$\beta_{\mathrm{U}} = \left(\frac{N!}{L! L^{N-L}} \right)^{\frac{1}{N}}. \tag{35}$$

Note that (35) is equal to the asymptotic penalty for large SNR, derived in (33) and, as we will see in the examples, it appears to be a tight upper bound to the SNR penalty. Figs. 1 and 2 show the SEP for coherent detection of 4-ary orthogonal and 8-ary biorthogonal signaling for various L and N. The SEPs are calculated using new analytical expressions for the error probability of these constellations derived in [8]. Also the SEP of MRC operating at SNR values scaled by β_{L} and $\beta_{\mathrm{U}}^{\mathrm{A}}$ are plotted for each value of L and N. One can see in Figs. 1 and 2 that the SEP of hybrid diversity, $P_{e,\text{H-S/MRC}}(\Gamma)$, is lower and upper bounded by $P_{e,\mathrm{MRC}}({\beta_{\mathrm{L}}}^{-1}\Gamma)$ and $P_{e,\mathrm{MRC}}({\beta_{\mathrm{L}}^{\mathrm{A}}}^{-1}\Gamma)$, respectively. Note also that as L approaches N the differences between the SEP of H-S/MRC, and the conjectured bounds become very small.

Figs. 3 and 4 illustrate the exact SNR penalty of 4-ary orthogonal signaling as a function of average SNR per branch for $(L, N) = (3, 4)$ and $(L, N) = (4, 6)$, respectively. Note that in Figs. 3 and 4 β_{L} and β_{U} appear as lower and upper bounds to the SNR penalty.

In Figs. 5 and 6, the exact SNR penalty of 8-ary biorthogonal signaling is plotted for $(L, N) = (4, 6)$ and $(L, N) = (4, 8)$, respectively. Comparing Figs. 3 and 5 indicates that although the *asymptotic* SNR penalties are independent of the modulation order and the modulation format and are functions of L and N only, the SNR penalty is, in general, a function of the average SNR. For example, the SNR penalty curves for 4-ary orthogonal and 8-ary biorthogonal signaling in Figs. 3 and 5 have different shapes.

VI. Conclusion

In this paper, we have established that the SNR penalties of H-S/MRC relative to MRC in Rayleigh fading have small

SNR and large SNR asymptotes. Analytical expressions for the asymptotes have been derived. The asymptotic SNR penalties depend only on L and N, and are independent of the modulation format, the modulation order, and the average SNR, for uncoded systems.

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