Fall 2006 - ME 582 Advanced Materials Science

Problem Solutions to HW#2

2.14 Stress-strain relationship -

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \cdot \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix}$$

Rearranging

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix}$$

For $\sigma_1 \neq 0$ while $\sigma_2 = \tau_{12} = 0$

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sigma_1 \\ 0 \\ 0 \end{pmatrix}$$

yields

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} Q_{22} \cdot Q_{66} \cdot \dfrac{\sigma_1}{\left(Q_{11} \cdot Q_{22} \cdot Q_{66} - Q_{12}^2 \cdot Q_{66}\right)} \\ Q_{12} \cdot Q_{66} \cdot \dfrac{\sigma_1}{\left(Q_{11} \cdot Q_{22} \cdot Q_{66} - Q_{12}^2 \cdot Q_{66}\right)} \\ 0 \end{pmatrix}$$

Longitudinal Young's modulus -

$$E_1 = \frac{\sigma_1}{\varepsilon_1}$$

Substituting and collecting terms, yields

$$E_1 = Q_{11} - \frac{Q_{12}^2}{Q_{22}}$$

Major Poisson's ratio -

$$\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1}$$

by substitution, yields

$$\nu_{12} = Q_{12} \cdot Q_{66} \cdot \frac{\sigma_1}{\left[\left(Q_{11} \cdot Q_{22} \cdot Q_{66} - Q_{12}^2 \cdot Q_{66}\right) \cdot \varepsilon_1\right]}$$

$$\nu_{12} = \frac{Q_{12}}{Q_{22}}$$

For $\sigma_2 \neq 0$ while $\sigma_1 = \tau_{12} = 0$

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ \sigma_2 \\ 0 \end{pmatrix}$$

yields

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} -Q_{12} \cdot Q_{66} \cdot \dfrac{\sigma_2}{\left(Q_{11} \cdot Q_{22} \cdot Q_{66} - Q_{12}^2 \cdot Q_{66}\right)} \\ Q_{11} \cdot Q_{66} \cdot \dfrac{\sigma_2}{\left(Q_{11} \cdot Q_{22} \cdot Q_{66} - Q_{12}^2 \cdot Q_{66}\right)} \\ 0 \end{bmatrix}$$

Transverse Young's modulus -

$$E_2 = \frac{\sigma_2}{\varepsilon_2}$$

Substituting and collecting terms, yields

$$E_2 = Q_{22} - \frac{Q_{12}^2}{Q_{11}}$$

Minor Poisson's ratio -

$$\nu_{21} = \frac{\varepsilon_1}{\varepsilon_2}$$

by substitution, yields

$$\nu_{21} = Q_{12} \cdot Q_{66} \cdot \frac{\sigma_2}{\left[\left(Q_{11} \cdot Q_{22} \cdot Q_{66} - Q_{12}^2 \cdot Q_{66}\right) \cdot \varepsilon_2\right]}$$

$$\nu_{21} = \frac{Q_{12}}{Q_{11}}$$

For $\tau_{12} \neq 0$ while $\sigma_1 = \sigma_2 = 0$

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix}^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ \tau_{12} \end{pmatrix}$$

yields

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{\tau_{12}}{Q_{66}} \end{bmatrix}$$

In-plane shear modulus -

$$G_{12} = \frac{\tau_{12}}{\gamma_{12}}$$

by substitution, yields

$$G_{12} = Q_{66}$$

2.18 Reduced stiffness and compliance matrices for a 60° angle lamina of Boron/Epoxy from Exercise 2.10 -

$$\mathbf{Q} = \begin{pmatrix} 204.98 & 4.28 & 0 \\ 4.28 & 18.59 & 0 \\ 0 & 0 & 5.59 \end{pmatrix} \cdot \mathrm{GPa}$$

$$\mathbf{S} = \begin{pmatrix} 4.902\cdot10^{-3} & -1.127\cdot10^{-3} & 0 \\ -1.127\cdot10^{-3} & 5.405\cdot10^{-2} & 0 \\ 0 & 0 & 1.789\cdot10^{-1} \end{pmatrix} \cdot \frac{1}{\mathrm{GPa}}$$

Transformed reduced stiffness matrix elements

$$\underline{Q}_{11} = Q_{11}\cdot c^4 + Q_{22}\cdot s^4 + 2\cdot(Q_{12} + 2\cdot Q_{66})\cdot s^2\cdot c^2 \qquad \underline{Q}_{11} = 29.06\cdot \mathrm{GPa}$$

$$\underline{Q}_{12} = (Q_{11} + Q_{22} - 4\cdot Q_{66})\cdot s^2\cdot c^2 + Q_{12}\cdot(c^4 + s^4) \qquad \underline{Q}_{12} = 40.40\cdot \mathrm{GPa}$$

$$\underline{Q}_{22} = Q_{11}\cdot s^4 + Q_{22}\cdot c^4 + 2\cdot(Q_{12} + 2\cdot Q_{66})\cdot s^2\cdot c^2 \qquad \underline{Q}_{22} = 122.3\cdot \mathrm{GPa}$$

$$\underline{Q}_{16} = (Q_{11} - Q_{12} - 2\cdot Q_{66})\cdot c^3\cdot s - (Q_{22} - Q_{12} - 2\cdot Q_{66})\cdot s^3\cdot c \qquad \underline{Q}_{16} = 19.50\cdot \mathrm{GPa}$$

$$\underline{Q}_{26} = (Q_{11} - Q_{12} - 2\cdot Q_{66})\cdot s^3\cdot c - (Q_{22} - Q_{12} - 2\cdot Q_{66})\cdot c^3\cdot s \qquad \underline{Q}_{26} = 61.21\cdot \mathrm{GPa}$$

$$\underline{Q}_{66} = (Q_{11} + Q_{22} - 2\cdot Q_{12} - 2\cdot Q_{66})\cdot s^2\cdot c^2 + Q_{66}\cdot(s^4 + c^4) \qquad \underline{Q}_{66} = 41.71\cdot \mathrm{GPa}$$

Transformed reduced stiffness matrix

$$\underline{Q} = \begin{pmatrix} \underline{Q}_{11} & \underline{Q}_{12} & \underline{Q}_{16} \\ \underline{Q}_{12} & \underline{Q}_{22} & \underline{Q}_{26} \\ \underline{Q}_{16} & \underline{Q}_{26} & \underline{Q}_{66} \end{pmatrix}$$

$$\underline{Q} = \begin{pmatrix} 29.06 & 40.40 & 19.50 \\ 40.40 & 122.26 & 61.21 \\ 19.50 & 61.21 & 41.71 \end{pmatrix} \cdot \mathrm{GPa}$$

Transformed compliance matrix elements

$$\underline{S}_{11} = S_{11}\cdot c^4 + (2\cdot S_{12} + S_{66})\cdot s^2\cdot c^2 + S_{22}\cdot s^4 \qquad \underline{S}_{11} = 6.383\cdot10^{-2}\cdot\frac{1}{\mathrm{GPa}}$$

$$\underline{S}_{12} = S_{12}\cdot(s^4 + c^4) + (S_{11} + S_{22} - S_{66})\cdot s^2\cdot c^2 \qquad \underline{S}_{12} = -2.319\cdot10^{-2}\cdot\frac{1}{\mathrm{GPa}}$$

$$\underline{S}_{22} = S_{11}\cdot s^4 + (2\cdot S_{12} + S_{66})\cdot s^2\cdot c^2 + S_{22}\cdot c^4 \qquad \underline{S}_{22} = 3.925\cdot10^{-2}\cdot\frac{1}{\mathrm{GPa}}$$

$$\underline{S}_{16} = (2\cdot S_{11} - 2\cdot S_{12} - S_{66})\cdot s\cdot c^3 - (2\cdot S_{22} - 2\cdot S_{12} - S_{66})\cdot s^3\cdot c \qquad \underline{S}_{16} = 4.195\cdot10^{-3}\cdot\frac{1}{\mathrm{GPa}}$$

$$\underline{S}_{26} = (2\cdot S_{11} - 2\cdot S_{12} - S_{66})\cdot c\cdot s^3 - (2\cdot S_{22} - 2\cdot S_{12} - S_{66})\cdot c^3\cdot s \qquad \underline{S}_{26} = -4.676\cdot10^{-2}\cdot\frac{1}{\mathrm{GPa}}$$

$$\underline{S}_{66} = 2\cdot(2\cdot S_{11} + 2\cdot S_{22} - 4\cdot S_{12} - S_{66})\cdot s^2\cdot c^2 + S_{66}\cdot(s^4 + c^4) \qquad \underline{S}_{66} = 9.063\cdot10^{-2}\cdot\frac{1}{\mathrm{GPa}}$$

Transformed compliance matrix-

$$\underline{S} = \begin{pmatrix} \underline{S}_{11} & \underline{S}_{12} & \underline{S}_{16} \\ \underline{S}_{12} & \underline{S}_{22} & \underline{S}_{26} \\ \underline{S}_{16} & \underline{S}_{26} & \underline{S}_{66} \end{pmatrix}$$

$$\underline{S} = \begin{pmatrix} 6.383\cdot10^{-2} & -2.319\cdot10^{-2} & 4.195\cdot10^{-3} \\ -2.319\cdot10^{-2} & 3.925\cdot10^{-2} & -4.676\cdot10^{-2} \\ 4.195\cdot10^{-3} & -4.676\cdot10^{-2} & 9.063\cdot10^{-2} \end{pmatrix} \cdot \frac{1}{\mathrm{GPa}}$$

2.20 Global stresses acting on a 60° lamina of Boron/Epoxy:

$\sigma_x := 4 \cdot MPa$ $\qquad \sigma_y := 2 \cdot MPa$ $\qquad \tau_{xy} := -3 \cdot MPa$

From Exercise 2.18

$$\underline{S} := \begin{pmatrix} 6.383 \cdot 10^{-2} & -2.319 \cdot 10^{-2} & 4.195 \cdot 10^{-3} \\ -2.319 \cdot 10^{-2} & 3.925 \cdot 10^{-2} & -4.676 \cdot 10^{-2} \\ 4.195 \cdot 10^{-3} & -4.676 \cdot 10^{-2} & 9.063 \cdot 10^{-2} \end{pmatrix} \cdot \frac{1}{GPa}$$

1) Global strains

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} := \underline{S} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} := \begin{pmatrix} 6.383 \cdot 10^{-2} & -2.319 \cdot 10^{-2} & 4.195 \cdot 10^{-3} \\ -2.319 \cdot 10^{-2} & 3.925 \cdot 10^{-2} & -4.676 \cdot 10^{-2} \\ 4.195 \cdot 10^{-3} & -4.676 \cdot 10^{-2} & 9.063 \cdot 10^{-2} \end{pmatrix} \cdot \frac{1}{GPa} \cdot \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \cdot MPa$$

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} 196.4 \\ 126.0 \\ -348.6 \end{pmatrix} \cdot \frac{\mu m}{m}$$

2) Local stresses

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} := \mathbf{T} \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} := \begin{pmatrix} 0.25 & 0.75 & 0.866 \\ 0.75 & 0.25 & -0.866 \\ -0.433 & 0.433 & -0.5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \cdot MPa$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} -9.800 \cdot 10^{-2} \\ 6.098 \\ 6.340 \cdot 10^{-1} \end{pmatrix} \cdot MPa$$

Local strains

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} := \mathbf{R} \cdot \mathbf{T} \cdot \mathbf{R}^{-1} \cdot \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} -7.36 \\ 329.73 \\ 113.40 \end{pmatrix} \cdot \frac{\mu m}{m}$$

3) Principal normal stresses produced by applied global stresses -

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}} \qquad \sigma_{max} = 6.162 \cdot MPa$$

$$\sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}} \qquad \sigma_{min} = -0.1623 \cdot MPa$$

Orientation of maximum principal stress -

$$\theta_{p\sigma} = \frac{1}{2} \cdot atan\left(\frac{2 \cdot \tau_{xy}}{\sigma_x - \sigma_y}\right) \cdot \left(\frac{180 \cdot °}{\pi}\right) \qquad \theta_{p\sigma} = -35.78 \cdot °$$

Principal normal strains -

$$\varepsilon_{max} = \frac{\varepsilon_x + \varepsilon_y}{2} + \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \varepsilon_{max} = 339.0 \cdot \frac{\mu m}{m}$$

$$\varepsilon_{min} = \frac{\varepsilon_x - \varepsilon_y}{2} - \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \varepsilon_{min} = -16.64 \cdot \frac{\mu m}{m}$$

Orientation of maximum principal strain -

$$\theta_{p\varepsilon} = \frac{1}{2} \cdot atan\left(\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}\right) \cdot \left(\frac{180 \cdot °}{\pi}\right) \qquad \theta_{p\varepsilon} = -39.30 \cdot °$$

d) Maximum shear produced by applied global stresses -

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\;2}} \qquad \tau_{max} = 3.162 \cdot MPa$$

Orientation of maximum shear -

$$\theta_{s\tau} = \frac{1}{2} \cdot atan\left(-\frac{\sigma_x - \sigma_y}{2 \cdot \tau_{xy}}\right) \cdot \left(\frac{180 \cdot °}{\pi}\right) \qquad \theta_{s\tau} = 9.22 \cdot °$$

Maximum shear strain -

$$\gamma_{max} = 2 \cdot \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \gamma_{max} = 355.7 \cdot \frac{\mu m}{m}$$

Orientation of maximum shear strain -

$$\theta_{s\gamma} = \frac{1}{2} \cdot atan\left(-\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}}\right) \cdot \left(\frac{180 \cdot °}{\pi}\right) \qquad \theta_{s\gamma} = 5.70 \cdot °$$

2.22 Elastic properties for a unidirectional Boron/Epoxy lamina from Table 2.2

Longitudinal Young's Modulus :	$E_1 := 29.59 \cdot Msi$
Transverse Young's Modulus :	$E_2 := 2.683 \cdot Msi$
Major Poisson's Ratio :	$\nu_{12} := 0.23$
Shear Modulus :	$G_{12} := 0.811 \cdot Msi$

Compliance matrix elements

$$S_{11} := \frac{1}{E_1} \qquad S_{11} = 3.380 \cdot 10^{-2} \cdot \frac{1}{Msi}$$

$$S_{22} := \frac{1}{E_2} \qquad S_{22} = 3.727 \cdot 10^{-1} \cdot \frac{1}{Msi}$$

$$S_{12} := -\frac{\nu_{12}}{E_1} \qquad S_{12} = -7.773 \cdot 10^{-3} \cdot \frac{1}{Msi}$$

$$S_{66} := \frac{1}{G_{12}} \qquad S_{66} = 1.233 \cdot \frac{1}{Msi}$$

Transformed compliance matrix elements

$$\underline{S}_{11} := S_{11} \cdot c^4 + (2 \cdot S_{12} + S_{66}) \cdot s^2 \cdot c^2 + S_{22} \cdot s^4 \qquad \underline{S}_{11} = 4.400 \cdot 10^{-1} \cdot \frac{1}{Msi}$$

$$\underline{S}_{12} := S_{12} \cdot (s^4 + c^4) + (S_{11} + S_{22} - S_{66}) \cdot s^2 \cdot c^2 \qquad \underline{S}_{12} = -1.598 \cdot 10^{-1} \cdot \frac{1}{Msi}$$

$$\underline{S}_{22} := S_{11} \cdot s^4 + (2 \cdot S_{12} + S_{66}) \cdot s^2 \cdot c^2 + S_{22} \cdot c^4 \qquad \underline{S}_{22} = 2.706 \cdot 10^{-1} \cdot \frac{1}{Msi}$$

$$\underline{S}_{16} := (2 \cdot S_{11} - 2 \cdot S_{12} - S_{66}) \cdot s \cdot c^3 - (2 \cdot S_{22} - 2 \cdot S_{12} - S_{66}) \cdot s^3 \cdot c \qquad \underline{S}_{16} = 2.883 \cdot 10^{-2} \cdot \frac{1}{Msi}$$

$$\underline{S}_{26} := (2 \cdot S_{11} - 2 \cdot S_{12} - S_{66}) \cdot c \cdot s^3 - (2 \cdot S_{22} - 2 \cdot S_{12} - S_{66}) \cdot c^3 \cdot s \qquad \underline{S}_{26} = -3.223 \cdot 10^{-1} \cdot \frac{1}{Msi}$$

$$\underline{S}_{66} := 2 \cdot (2 \cdot S_{11} + 2 \cdot S_{22} - 4 \cdot S_{12} - S_{66}) \cdot s^2 \cdot c^2 + S_{66} \cdot (s^4 + c^4) \qquad \underline{S}_{66} = 6.248 \cdot 10^{-1} \cdot \frac{1}{Msi}$$

Engineering constants

$$E_x := \frac{1}{\underline{S}_{11}} \qquad E_x = 2.272 \cdot Msi$$

$$\nu_{xy} := -\frac{\underline{S}_{12}}{\underline{S}_{11}} \qquad \nu_{xy} = 0.3632$$

$$m_x := -\underline{S}_{16} \cdot E_1 \qquad m_x = -0.8530$$

$$E_y := \frac{1}{\underline{S}_{22}} \qquad E_y = 3.696 \cdot Msi$$

$$m_y := -\underline{S}_{26} \cdot E_1 \qquad m_y = 9.538$$

$$G_{xy} := \frac{1}{\underline{S}_{66}} \qquad G_{xy} = 1.6 \cdot Msi$$