ME 582 Advanced Materials Science

Chapter 2 Macromechanical Analysis of a Lamina (Part 1)

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HW #2

- 2.14
- 2.18
- 2.20
- 2.22

Drop your solution to Dr. Jan Gou's mailbox by 5:00 PM, 09/20/2006, Wednesday.

Lamina and Laminate

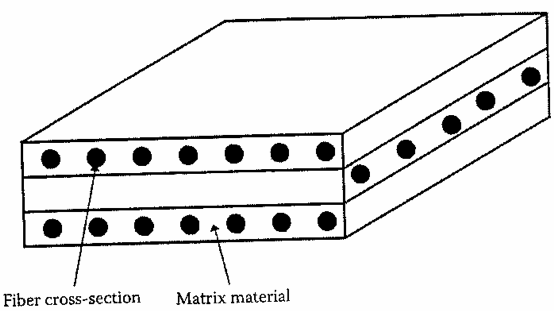

FIGURE 2.1
Typical laminate made of three laminae.

General Material in 3D Stress State

- Stiffness matrix [C]

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}, \quad (2.25)$$

Stiffness matrix [C] has 36 constants

General Material in 3D Stress State

- Compliance matrix [S]

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}. \qquad (2.26)$$

Isotropic Material in 3D Stress State

- If all planes in an orthotropic body are identical, it is an isotropic material
- Homogeneous and linearly elastic material
- Compliance matrix [S]

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}, \qquad (2.17)$$

Isotropic Material in 3D Stress State

- Stiffness matrix [C]

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} =
\begin{bmatrix}
\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
0 & 0 & 0 & G & 0 & 0 \\
0 & 0 & 0 & 0 & G & 0 \\
0 & 0 & 0 & 0 & 0 & G
\end{bmatrix}
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix}, \qquad (2.18)
$$

Plane Stress Assumption

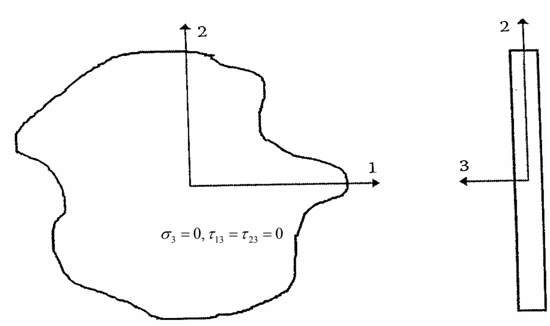

FIGURE 2.17
Plane stress conditions for a thin plate.

- Upper and lower surfaces are free from external loads
- $\sigma_3 = 0, \tau_{13} = \tau_{23} = 0$

  $\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2, \gamma_{23} = \gamma_{31} = 0$

Reduction of Hooke's Law in 3D to 2D

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}. \qquad (2.26)$$

- Reduction of compliance matrix

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Reduction of Hooke's Law in 3D to 2D

- Reduction of stiffness matrix

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2},$$

$$Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2}, \qquad (2.79a\text{–}d)$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2},$$

$$Q_{66} = \frac{1}{S_{66}}.$$

Compliance and Stiffness Matrix vs. Elastic Constants

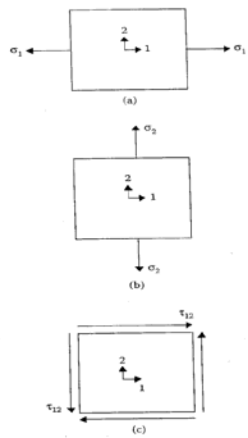

FIGURE 2.18
Application of stresses to find engineering constants of a unidirectional lamina.

- (a) Pure tensile load in direction 1
- (b) Pure tensile load in direction 2
- (c) Pure shear stress in the plane of 1-2

Pure Load in Direction 1

- Apply a pure tensile load in direction 1 (Figure 2.18a), that is,

$$\sigma_1 \neq 0,\ \sigma_2 = 0,\ \tau_{12} = 0. \qquad (2.80)$$

$$\varepsilon_1 = S_{11}\sigma_1,$$
$$\varepsilon_2 = S_{12}\sigma_1, \qquad (2.81a\text{–}c)$$
$$\gamma_{12} = 0.$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$E_1 \equiv \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}}, \qquad (2.82)$$

$$\nu_{12} \equiv -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}}. \qquad (2.83)$$

Pure Load in Direction 2

Apply a pure tensile load in direction 2 (Figure 2.18b), that is

$$\sigma_1 = 0,\ \sigma_2 \neq 0,\ \tau_{12} = 0. \quad (2.84)$$

$$\varepsilon_1 = S_{12}\sigma_2,$$
$$\varepsilon_2 = S_{22}\sigma_2, \quad (2.85a–c)$$
$$\gamma_{12} = 0.$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$E_2 \equiv \frac{\sigma_2}{\varepsilon_2} = \frac{1}{S_{22}}, \quad (2.86)$$

$$\nu_{21} \equiv -\frac{\varepsilon_1}{\varepsilon_2} = -\frac{S_{12}}{S_{22}}. \quad (2.87)$$

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}. \quad (2.88)$$

Pure Shear Stress in the Plane 1-2

Apply a pure shear stress in the plane 1–2 (Figure 2.18c) — that is,

$$\sigma_1 = 0,\ \sigma_2 = 0 \text{ and } \tau_{12} \neq 0. \quad (2.89)$$

$$\varepsilon_1 = 0,$$

$$\varepsilon_2 = 0,$$

$$\gamma_{12} = S_{66}\tau_{12}. \quad (2.90a–c)$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$G_{12} \equiv \frac{\tau_{12}}{\gamma_{12}} = \frac{1}{S_{66}}. \quad (2.91)$$

Coefficients of Compliance Matrix

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2},$$

$$S_{66} = \frac{1}{G_{12}}. \qquad (2.92a–d)$$

Coefficients of Stiffness Matrix

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2},$$

$$Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2}, \qquad (2.79a–d)$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2},$$

$$Q_{66} = \frac{1}{S_{66}}.$$

$$Q_{11} = \frac{E_1}{1 - \nu_{21}\nu_{12}},$$

$$Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{21}\nu_{12}},$$

$$Q_{22} = \frac{E_2}{1 - \nu_{21}\nu_{12}}, \text{ and}$$

$$Q_{66} = G_{12}. \qquad (2.93a–d)$$

Hooke's Law for a 2D Angle Lamina

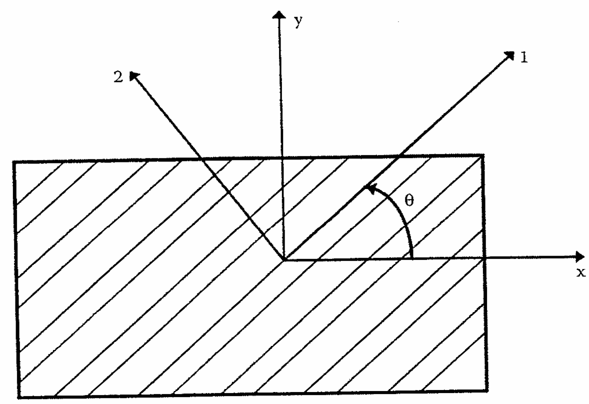

FIGURE 2.20
Local and global axes of an angle lamina.

Global and Local Stresses

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}, \qquad (2.94)$$

$$[T]^{-1} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix}, \qquad (2.95)$$

$$[T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix}, \qquad (2.96)$$

$$c = \text{Cos}\,(\theta),$$

$$s = \text{Sin}\,(\theta). \qquad (2.97a,b)$$

Global and Local Strains

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = [R][T][R]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \qquad (2.100)$$

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \qquad (2.101)$$

Global Stress and Strain

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1}[Q] \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}. \qquad (2.98)$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1}[Q][R][T][R]^{-1} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}. \qquad (2.102)$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \qquad (2.103)$$

Transformed Reduced Stiffness Matrix

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \qquad (2.103)$$

- Elements of transformed reduced stiffness matrix:

$$\bar{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^2),$$

$$\bar{Q}_{22} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\bar{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})s^3c,$$

$$\bar{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s,$$

$$\bar{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4). \qquad (2.104a–f)$$

Transformed Reduced Compliance Matrix

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \qquad (2.105)$$

- Elements of transformed reduced compliance matrix:

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4,$$

$$\bar{S}_{12} = S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c,$$

$$\bar{S}_{26} = (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4). \qquad (2.106a–f)$$

Engineering Constants of an Angle Lamina

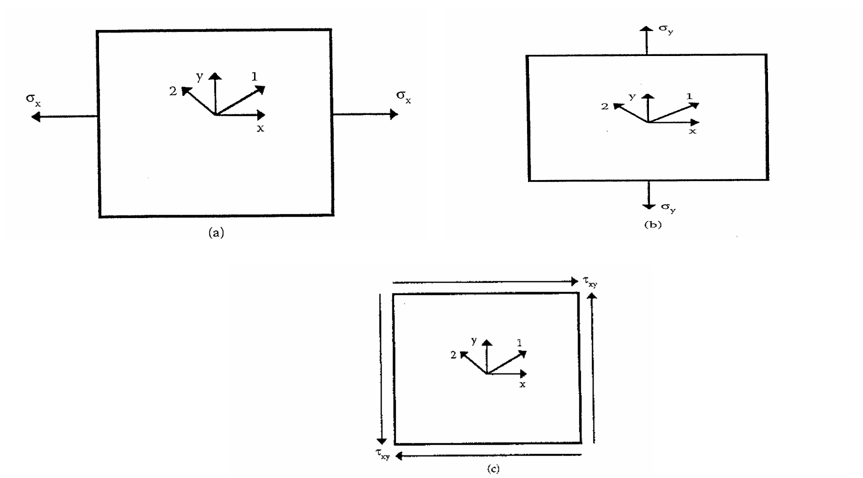

FIGURE 2.23
Application of stresses to find engineering constants of an angle lamina.

Pure Load in Direction *x*

$$\sigma_x \neq 0,\ \sigma_y = 0,\ \tau_{xy} = 0. \qquad (2.115)$$

$$\varepsilon_x = \bar{S}_{11}\sigma_x,$$

$$\varepsilon_y = \bar{S}_{12}\sigma_x,$$

$$\gamma_{xy} = \bar{S}_{16}\sigma_x\ . \qquad (2.116a–c)$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \qquad (2.105)$$

$$E_x \equiv \frac{\sigma_x}{\varepsilon_x} = \frac{1}{\bar{S}_{11}}. \qquad (2.117)$$

$$\nu_{xy} \equiv -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\bar{S}_{12}}{\bar{S}_{11}}. \qquad (2.118)$$

Shear Coupling Term m_x

- In an angle lamina, interaction occurs between the shear strain and the normal stress.
- The shear coupling term that relates the normal stress in the *x*-direction to the shear strain is denoted by m_x
- The same parameter, m_x, relates the shear stress in the *x-y* plane to the normal strain in direction-*x*
- m_x is a non-dimensional parameter like Poisson's ratio

$$\frac{1}{m_x} = -\frac{1}{\bar{S}_{16} E_1}$$

Pure Load in Direction *y*

$$\sigma_x = 0,\ \sigma_y \neq 0,\ \tau_{xy} = 0\ , \qquad (2.120)$$

$$E_y = \frac{1}{\bar{S}_{22}}, \qquad (2.121)$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \qquad (2.105)$$

$$\nu_{yx} = -\frac{\bar{S}_{12}}{\bar{S}_{22}}, \text{ and} \qquad (2.122)$$

$$\frac{1}{m_y} = -\frac{1}{\bar{S}_{26} E_1}. \qquad (2.123)$$

$$\frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x}. \qquad (2.124)$$

Shear Coupling Term m_y

- In an angle lamina, interaction occurs between the shear strain and the normal stress.
- The shear coupling term that relates the normal stress in the y-direction to the shear strain is denoted by m_y
- The same parameter, m_y, relates the shear stress in the *x-y* plane to the normal strain in direction-y
- m_y is a non-dimensional parameter like Poisson's ratio

$$\frac{1}{m_y} = -\frac{1}{\bar{S}_{26} E_1}$$

Shear in the *x-y* Plane

$$\sigma_x = 0,\ \sigma_y = 0,\ \tau_{xy} \neq 0\ , \qquad (2.125)$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \qquad (2.105)$$

$$\frac{1}{m_x} = -\frac{1}{\bar{S}_{16} E_1}, \qquad (2.126)$$

$$\frac{1}{m_y} = -\frac{1}{\bar{S}_{26} E_1}, \text{ and} \qquad (2.127)$$

$$G_{xy} = \frac{1}{\bar{S}_{66}}. \qquad (2.128)$$

Strain-Stress Equation of an Angle Lamina

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \qquad (2.105)$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{m_x}{E_1} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{m_y}{E_1} \\ -\frac{m_x}{E_1} & -\frac{m_y}{E_1} & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}. \qquad (2.129)$$

Engineering Constants of an Angle Lamina

$$\frac{1}{E} = \bar{S}_{11}$$

$$= S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4.$$

$$= \frac{1}{E_1}c^4 + \left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_1}\right)s^2c^2 + \frac{1}{E_2}s^4, \qquad (2.130)$$

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4,$$

$$S_{12} = S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c,$$

$$S_{26} = (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4). \qquad (2.106a–f)$$

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2},$$

$$S_{66} = \frac{1}{G_{12}}. \qquad (2.92a–d)$$

Engineering Constants of an Angle Lamina

$$\nu_{xy} = -E_x \bar{S}_{12}$$

$$= -E_x[S_{12}(s^4+c^4) + (S_{11}+S_{22}-S_{66})s^2c^2]$$

$$= E_x\left[\frac{\nu_{12}}{E_1}(s^4+c^4) - \left(\frac{1}{E_1}+\frac{1}{E_2}-\frac{1}{G_{12}}\right)s^2c^2\right], \quad (2.131)$$

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12}+S_{66})s^2c^2 + S_{22}s^4,$$

$$\bar{S}_{12} = S_{12}(s^4+c^4) + (S_{11}+S_{22}-S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12}+S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11}-2S_{12}-S_{66})sc^3 - (2S_{22}-2S_{12}-S_{66})s^3c,$$

$$\bar{S}_{26} = (2S_{11}-2S_{12}-S_{66})s^3c - (2S_{22}-2S_{12}-S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11}+2S_{22}-4S_{12}-S_{66})s^2c^2 + S_{66}(s^4+c^4). \quad (2.106a\text{–}f)$$

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2},$$

$$S_{66} = \frac{1}{G_{12}}. \quad (2.92a\text{–}d)$$

Engineering Constants of an Angle Lamina

$$\frac{1}{E_y} = \bar{S}_{22}$$

$$= S_{11}s^4 + (2S_{12}+S_{66})c^2s^2 + S_{22}c^4$$

$$= \frac{1}{E_1}s^4 + \left(-\frac{2\nu_{12}}{E_1}+\frac{1}{G_{12}}\right)c^2s^2 + \frac{1}{E_2}c^4, \quad (2.132)$$

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12}+S_{66})s^2c^2 + S_{22}s^4,$$

$$\bar{S}_{12} = S_{12}(s^4+c^4) + (S_{11}+S_{22}-S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12}+S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11}-2S_{12}-S_{66})sc^3 - (2S_{22}-2S_{12}-S_{66})s^3c,$$

$$\bar{S}_{26} = (2S_{11}-2S_{12}-S_{66})s^3c - (2S_{22}-2S_{12}-S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11}+2S_{22}-4S_{12}-S_{66})s^2c^2 + S_{66}(s^4+c^4). \quad (2.106a\text{–}f)$$

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2},$$

$$S_{66} = \frac{1}{G_{12}}. \quad (2.92a\text{–}d)$$

Engineering Constants of an Angle Lamina

$$\frac{1}{G_{xy}} = \bar{S}_{66}$$

$$= 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4)$$

$$= 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^2c^2 + \frac{1}{G_{12}}(s^4 + c^4), \qquad (2.133)$$

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4,$$

$$\bar{S}_{12} = S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c,$$

$$\bar{S}_{26} = (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4). \qquad (2.106a\text{–}f)$$

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2},$$

$$S_{66} = \frac{1}{G_{12}}. \qquad (2.92a\text{–}d)$$

Engineering Constants of an Angle Lamina

$$m_x = -\bar{S}_{16}E_1$$

$$= -E_1[(S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c]$$

$$= E_1\left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)sc^3 + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^3c\right], \qquad (2.134)$$

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4,$$

$$\bar{S}_{12} = S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c,$$

$$\bar{S}_{26} = (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4). \qquad (2.106a\text{–}f)$$

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2},$$

$$S_{66} = \frac{1}{G_{12}}. \qquad (2.92a\text{–}d)$$

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Engineering Constants of an Angle Lamina

$$m_y = -\bar{S}_{26} E_1$$

$$= -E_1[(2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3]$$

$$= E_1\left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)s^3c + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)sc^3\right]. \qquad (2.135)$$

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4,$$

$$S_{12} = S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c,$$

$$S_{26} = (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4). \qquad (2.106a–f)$$

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2},$$

$$S_{66} = \frac{1}{G_{12}}. \qquad (2.92a–d)$$

Invariant Form of Stiffness Matrix

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}, \qquad (2.103)$$

$$\bar{Q}_{11} = U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta,$$

$$\bar{Q}_{12} = U_4 - U_3 \cos 4\theta,$$

$$\bar{Q}_{22} = U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta$$

$$\bar{Q}_{16} = \frac{U_2}{2} \sin 2\theta + U_3 \sin 4\theta,$$

$$\bar{Q}_{26} = \frac{U_2}{2} \sin 2\theta - U_3 \sin 4\theta,$$

$$\bar{Q}_{66} = \frac{1}{2}(U_1 - U_4) - U_3 \cos 4\theta, \qquad (2.137a–f)$$

Invariant Form of Stiffness Matrix

$$U_1 = \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})$$

$$U_2 = \frac{1}{2}(Q_{11} - Q_{22}),$$

$$U_3 = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}),$$

$$U_4 = \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}). \qquad (2.138a–d)$$

$$Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{21}\nu_{12}},$$

$$Q_{22} = \frac{E_2}{1 - \nu_{21}\nu_{12}}, \text{ and}$$

$$Q_{66} = G_{12}. \qquad (2.93a–d)$$

Invariant Form of Compliance Matrix

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \qquad (2.105)$$

$$\bar{S}_{11} = V_1 + V_2 \operatorname{Cos} 2\theta + V_3 \operatorname{Cos} 4\theta,$$

$$\bar{S}_{12} = V_4 - V_3 \operatorname{Cos} 4\theta,$$

$$\bar{S}_{22} = V_1 - V_2 \operatorname{Cos} 2\theta + V_3 \operatorname{Cos} 4\theta,$$

$$\bar{S}_{16} = V_2 \operatorname{Sin} 2\theta + 2V_3 \operatorname{Sin} 4\theta,$$

$$\bar{S}_{26} = V_2 \operatorname{Sin} 2\theta - 2V_3 \operatorname{Sin} 4\theta, \text{ and}$$

$$\bar{S}_{66} = 2(V_1 - V_4) - 4V_3 \operatorname{Cos} 4\theta, \qquad (2.139a–f)$$

Invariant Form of Compliance Matrix

$$V_1 = \frac{1}{8}(3S_{11} + 3S_{22} + 2S_{12} + S_{66}),$$

$$V_2 = \frac{1}{2}(S_{11} - S_{22}),$$

$$V_3 = \frac{1}{8}(S_{11} + S_{22} - 2S_{12} - S_{66}),$$

$$V_4 = \frac{1}{8}(S_{11} + S_{22} + 6S_{12} - S_{66}). \quad (2.140a–d)$$

$$S_{11} = \frac{1}{E_1},$$

$$S_{12} = -\frac{\nu_{12}}{E_1},$$

$$S_{22} = \frac{1}{E_2},$$

$$S_{66} = \frac{1}{G_{12}}. \quad (2.92a–d)$$