Math 518 Spring 2002 NAME:

Do 5 of the following 8 problems

1. Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 3 & 6 \\ 2 & 2 & 0 & 1 & 7 \\ -1 & 1 & -2 & 0 & 1 \\ 4 & 1 & 3 & 0 & 6 \end{bmatrix}$ with reduced echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis of the null space and a basis for the range of the linear transformation $T(v) = \mathbf{A}v$.
(b) Is the linear transformation T one-to-one? onto?

2. Let $S = \{v_1, v_2, ..., v_n\}$ be a set of vectors in a vector space V. Show that S is linearly independent if and only if every vector in the span of S can be represented **uniquely** as a linear combination of the elements in S.

3. Let V be a vector space, W a subspace of V, and $T \in \mathcal{L}(V, V)$. Define the subset $T^{-1}(W) = \{v \in V \mid T(v) \in W\}$. Show that $T^{-1}(W)$ is a subspace of V.

4. Let $\mathbf{P}_4(\mathbf{R})$ denote the space of all polynomials in x of degree less than or equal to 4 with coefficients in $\mathbf{R}$. Define the linear map $D : \mathbf{P}_4(\mathbf{R}) \to \mathbf{P}_4(\mathbf{R})$ as $D(p(x)) = p'(x)$. Find all eigenvalues of D then choose a basis $\mathcal{B}$ of generalized eigenvectors and find the matrix of D corresponding to $\mathcal{B}$.

5. Let V be a finite dimensional vector space and $T \in \mathcal{L}(V, V)$. Suppose that $\text{range}(T) = \text{range}(T^2)$. Prove that $V = \text{range}(T) \oplus \text{null}(T)$.

6. (a) Prove or disprove that the following is a linear transformation:
$T : \mathcal{C}^\infty \to \mathcal{C}^\infty$ with $T(f) = x \cdot f'$.
(b) Find the null space of T.

7. Define $S : \mathbf{P}_4(\mathbf{R}) \to \mathbf{P}_4(\mathbf{R})$ as $S(p(x)) = p(x - a)$, with $a \in \mathbf{R}$. Find the matrix of S with respect to the standard basis $\{1, x, x^2, x^3, x^4\}$.

8. Let V be a finite dimensional vector space, $T \in \mathcal{L}(V)$, and $v \in V$. Let p denote the monic polynomial of smallest degree such that $p(T)(v) = 0$. Prove that p divides the minimal polynomial of T.