

## Math 518 review for exam 1 Spring 2002

1 The matrix  $\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 2 & 6 & -2 & 4 & 18 \\ 0 & 1 & 1 & 3 & 10 \\ 3 & 10 & -3 & 7 & 31 \end{bmatrix}$  has reduced echelon form  $\begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 4 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find a basis of the null space, a basis for the row space, and a basis for the image of the linear transformation  $T(X) = \mathbf{A}X$ .

(b) Is the linear transformation  $T(X) = \mathbf{A}X$  one-to-one? onto?

2 Prove or disprove: The following set of vectors is linearly independent:  $\left\{ \begin{bmatrix} 3 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \\ -3 \end{bmatrix} \right\}$

3 For each of the following sets of functions, decide whether the set of functions is linearly dependent or linearly independent (rigorous proof):

(a)  $\{1, x, x^2, x^3\}$ . (b)  $\{\sin^2(x), \cos^2(x), \cos(2x), \tan(x), \ln(x^2 + 1)\}$ . (c)  $\{\ln x, x \ln x, x^2 \ln x\}$ .

4 Prove or disprove for each of the following that they are subspaces of  $\mathcal{C}^\infty$ , the space of functions from  $\mathbf{R} \rightarrow \mathbf{R}$  that have all derivatives.

(a)  $\{f \in \mathcal{C}^\infty \mid f'(\pi) = 0\}$ . (b)  $\{f \in \mathcal{C}^\infty \mid \int_{-1}^1 \sin(t)f(t)dt = 0\}$ . (c)  $\{f \in \mathcal{C}^\infty \mid \int_0^1 f^2(t)dt = 0\}$ .

(d)  $\{f \in \mathcal{C}^\infty \mid f'' + 2\lambda f' + \lambda^2 f = e^t\}$ . (e)  $\{f \in \mathcal{C}^\infty \mid \lim_{|t| \rightarrow \infty} e^{t^2} f = 0\}$ .

5 Prove that the intersection of two vector spaces is a subspace. Show that the union of two subspaces is a subspace if and only if one space is contained in the other.

6 Show that a linear transformation is one-to-one if and only if the null space is the zero space.

7 Prove or disprove for each of the following that it is a linear transformation:

(a)  $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$  with  $T(x, y) = (x - y, 1 - y + x)$ .

(b)  $T : \mathcal{C}^\infty \rightarrow \mathcal{C}^\infty$  with  $T(f) = f'$ .

(c)  $T : \mathcal{C}^\infty \rightarrow \mathcal{C}^\infty$  where  $T(f)$  is the antiderivative  $F$  of  $f$  that satisfies  $F(0) = 1$ .

(d)  $T : \mathcal{C}^\infty \rightarrow \mathcal{C}^\infty$  where  $T(f)$  is the antiderivative  $F$  of  $f$  that satisfies  $F(0) = 0$ .

(e)  $T : \mathcal{C}^\infty \rightarrow \mathcal{C}^\infty$  with  $T(f) = f(x + 1)$ .

8 Prove that  $\mathcal{U} = \{(x, y, z) \in \mathbf{R}^3 \mid 2x - 3y + z = 0\}$  is a subspace. Find a basis.

9 Suppose that  $V$  is finite dimensional and  $S, T \in \mathcal{L}(V)$ .

(a) Prove that  $ST$  is invertible if and only if  $S$  and  $T$  are invertible.

(b) Prove that  $ST$  and  $TS$  have the same eigenvalues.

10 Let  $T \in \mathcal{L}(\mathbf{C}^3)$  with  $T(x, y, z) = (y, z, 0)$ . Find all eigenvalues of  $T$ . Find a basis for  $\mathbf{C}^3$  that consists entirely of generalized eigenvectors and find the corresponding matrix.

11 Let  $T \in \mathcal{L}(V)$ ,  $m$  a positive integer,  $v \in V$  such that  $T^{m-1}(v) \neq 0$  but  $T^m(v) = 0$ . Prove that  $\{v, T(v), T^2(v), \dots, T^{m-1}(v)\}$  is linearly independent.

12 Let  $T \in \mathcal{L}(V)$ . Prove that  $V = \text{null}(T^n) \oplus \text{range}(T^n)$ , where  $n = \dim V$ .

13 Let  $T \in \mathcal{L}(V)$  such that  $T^m = 0$  for some integer  $m$ . Show that the only eigenvector of  $T$  is 0.

14 Let  $T \in \mathcal{L}(V)$  and  $v \in V$ . Let  $p$  be the monic polynomial of smallest degree such that  $p(T)(v) = 0$ . Prove that  $p$  divides the minimal polynomial of  $T$ .

15 Previous homework assignments.