Math 518 review for Exam 2 Spring 2002

1. Let $T \in \mathcal{L}(V)$, m a positive integer, $v \in V$ such that $T^{m-1}(v) \neq 0$ but $T^m(v) = 0$. Prove that $\{v, T(v), T^2(v), ..., T^{m-1}(v)\}$ is linearly independent.
2. Let V be a finite dimensional vector space and $T \in \mathcal{L}(V, V)$. Suppose that range$(T) =$ range(T^2). Prove that $V = \text{range}(T) \oplus \text{null}(T)$.
3. Let V be a finite dimensional vector space, $T \in \mathcal{L}(V)$, and $v \in V$. Let p denote the monic polynomial of smallest degree such that $p(T)(v) = 0$. Prove that p divides the minimal polynomial of T.
4. Let V be a finite dimensional vector space, $T, S \in \mathcal{L}(V)$. show that ST is nilpotent if and only if TS is nilpotent.
5. Prove that similar matrices have the same characteristic polynomial and the same minimal polynomial.
6. Let V be a finite dimensional vector space, $T \in \mathcal{L}(V)$ with $T^m = 0$ for some positive integer M. Show that the only eigenvalue of T is zero.
7. Let V be a finite dimensional vector space, $T \in \mathcal{L}(V)$ such that $T^2 = I$. Show that $V = V_+ \oplus V_-$, where $V_+ = \{v \in V \mid T(v) = v\}$ and $V_- = \{v \in V \mid T(v) = -v\}$.
8. Let V be a finite dimensional vector space, $T \in \mathcal{L}(V)$. If $T^2 = T$ we say T is a projection. If $T^2 = I$ we say T is an involution. Find the minimal polynomials of all projections and all involutions.
9. Let $\mathbf{P}_n(\mathbf{C})$ denote the vector space of polynomials in x with coefficients in $\mathbf{C}$ and degree $\leq n$.
(a) Define $D : \mathbf{P}_n(\mathbf{C}) \to \mathbf{P}_n(\mathbf{C})$ as $D(p(x)) = p'(x)$. Find the minimal polynomial of D.
(b) Define $S : \mathbf{P}_n(\mathbf{C}) \to \mathbf{P}_n(\mathbf{C})$ as $S(p(x)) = p(x+1)$. Find the minimal polynomial of S.
10. Let V be a finite dimensional vector space of $n \times n$ matrices and let $T \in \mathcal{L}(V)$ be defined as $T(A) = A^t$ (T maps any $n \times n$ matrix to its transpose). Show that the only eigenvalues are ± 1 and describe the corresponding eigenspaces.
11. Let $T \in \mathcal{L}(V)$ have characteristic polynomial $p(x) = (x-1)^5(x+4)^4x^2$ and minimal polynomial $m(x) = (x-1)^2(x+4)^3x$. Find all possible non-similar Jordan canonical forms of T.
12. Classify up to similarity all complex 3×3 matrices A with $A^3 = I$.
13. For each of the following matrices $\mathbf{A}$, find $e^{t\mathbf{A}}$.

(a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 & -1 \\ 1 & 2 & -2 \\ 0 & 2 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

14. Solve the following systems of differential equations:

$$(a) \begin{array}{rlrl} y_1'(t) = & 2y_1(t) - y_2(t) & y_1(0) = & 0 \\ y_2'(t) = & y_1(t) & y_2(0) = & 1 \end{array}.$$

$$(b) \quad y''(t) = 2y' - y \qquad y(0) = 0 \qquad y'(0) = 1.$$

15. Previous homework assignments.